# Theoretical and methodological issues in the study of conceptual and procedural knowledge: Reflections on a series of studies on Greek secondary students' knowledge of fractions

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## ABSTRACT

*In this article we present an overview of four studies investigating Greek secondary students' conceptual and procedural knowledge of fractions. We discuss the problem of defining conceptual and procedural knowledge, and the implications of adopting one particular definition over others. We draw on the studies and their results to discuss the problem of measuring conceptual and procedural knowledge; the issue of inter-individual differences in conceptual and procedural fraction knowledge; and related educational implications.*

## KEYWORDS

*Conceptual knowledge, procedural knowledge, fractions, rational numbers, secondary students*

## RÉSUMÉ

*Dans cet article, nous présentons un aperçu de quatre études examinant la connaissance conceptuelle ainsi que la connaissance procédurale des élèves du collège grec sur les fractions. Nous nous référons au problème lié à la définition de la connaissance conceptuelle et de la connaissance procédurale ainsi qu'aux conséquences de l'adoption d'une définition particulière parmi d'autres. Nous nous fondons sur les études et leurs résultats pour soulever le problème portant sur la mesure de la connaissance conceptuelle et de la connaissance procédurale, la question des différences interindividuelles entre la connaissance conceptuelle et la connaissance procédurale sur les fractions et les implications pour l'éducation.*

## MOTS-CLÉS

*Connaissance conceptuelle, connaissance procédurale, fractions, nombres rationnels, apprenants du collège grec*

## THEORETICAL FRAMEWORK[1]

Mathematics education researchers generally agree that mathematical competence critically depends on conceptual understanding as well as on procedural skill (De Corte, 2004; Kilpatrick, Swafford & Findell, 2001). Notwithstanding this widespread agreement, theorizing about "conceptual understanding" and "procedural skill" as well as about their relations has given rise to a still ongoing debate that touches on issues relevant to education as well as to cognition.

The distinction between conceptual and procedural knowledge has been instrumental in framing this problem (Rittle-Johnson, Siegler & Alibali, 2001; Star & Stylianides, 2013). In this article we present an overview of four studies investigating Greek secondary students' conceptual and procedural knowledge of fractions. We draw on these studies to discuss theoretical and methodological issues in this research area, and to highlight relevant educational implications.

### *Conceptual and procedural knowledge: Definitional issues*

The use of the terms "conceptual knowledge" and "procedural knowledge" gained prominence in the field of mathematics education following a seminal publication by Hiebert (1996). Looking at how these terms were used over the years, Rittle-Johnson and Schneider (2015) suggested that there is general consensus that conceptual knowledge should be defined as knowledge of concepts pertaining to a domain. In this definition we interpret "concepts" as "categories" (e.g., entities, processes, relations) (Crooks & Alibali, 2014). This is compatible with perspectives on understanding that are prominent in the field of mathematics education. For example, Kilpatrick, and colleagues (2001) refer to conceptual understanding in mathematics as "comprehension of mathematical concepts, operations, and relations" (p. 5). Taking fractions as a paradigmatic case, we suggest that conceptual knowledge about fractions necessarily involves knowledge about how these mathematical entities are represented (symbolically or otherwise) and encompasses knowledge of their features and properties (e.g., fractions have magnitudes, they are densely ordered); the mathematical relations in which fractions are involved (e.g., equivalence, order) as well as their relation to other mathematical entities (e.g., to natural numbers); the meaning of fraction operations and the principles to which they obey (e.g., the laws of arithmetic); and knowledge of their representational meaning (Nunes & Bryant, 2015), that is, their function as representations of quantities and relations between quantities. Even if not exhaustive, this list clearly shows that conceptual knowledge is a multi-facet construct.

On the other hand, procedural knowledge is defined as knowledge of sequences of actions taken to accomplish a specific goal, namely procedures (Rittle-Johnson & Schneider, 2015). All school-taught algorithms regarding fraction comparison or operations, for instance, qualify as procedures; thus the ability to carry them out would imply procedural knowledge.

This definition has been contested on the grounds that it depicts a very narrow picture for procedural knowledge as something that one either has, or does not have, ignoring its different qualities (Star, 2005; Star & Stylianides, 2013). Consider, for example, two students who are both apt to perform fraction addition, but only one of them recognizes when this operation can be appropriately applied. This student's procedural knowledge is arguably qualitatively different. Mathematics educators have taken this point into consideration. For example, Kilpatrick and colleagues (2001) used the term "procedural fluency" to delineate a component of mathematical proficiency that is valuable from an educational perspective and goes beyond "procedural

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[1] A shorter version of this paper was presented at the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11) and will be included in the proceedings.

knowledge". They defined procedural fluency as "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (p. 5). Procedural fluency encompasses procedural knowledge. One might argue, however, that conceptual knowledge is a prerequisite of procedural fluency.

Still, the definition of procedural knowledge as knowledge of procedures is endorsed by numerous studies in this research area (Rittle-Johnson & Schneider, 2015) and is valuable, because it makes a sharper distinction between conceptual and procedural knowledge, which in turn makes measuring these constructs separately possible. In addition, using this definition it is possible to capture the phenomenon of procedural skill without understanding that has long tantalized mathematics education.

### *Relations between conceptual and procedural knowledge*

The relation between conceptual and procedural knowledge is difficult to establish empirically. This is because the two types of knowledge are typically entangled and cannot always be separated (Rittle-Johnson et al., 2001). The first issue of interest in the field was the order of acquisition of the two types of knowledge (i.e., which type develops first). Two opposing theoretical perspectives emerged. According to the first, conceptual knowledge precedes procedural knowledge: Children first develop conceptual knowledge in a domain, and then use this knowledge to generate strategies and select procedures to solve problems. According to the second, children extract principles pertaining to a domain through repeated experiences with problem-solving in a domain. There was supporting evidence for both positions. For example, Gelman & Meck (1993) showed that 3 to 5-year-old children's implicit knowledge of the counting principles preceded their ability to count accurately. On the other hand, Siegler & Stern (1998) showed that second graders (mean age 8.75 years) developed a shortcut strategy to compute sums of the type a+b-b (relying on the inversion principle) after they were exposed to problem solving experiences. Such contradictory evidence may be due to assumptions as to what counts as conceptual or procedural knowledge and also to methodological limitations of the studies (Rittle-Johnson et al., 2001). However, considering the two examples above, one cannot ignore the fact that there is a difference between the development of either type of knowledge that occurs spontaneously in the course of cognitive development within western sociocultural contexts, and the development that is instruction-induced (Inagaki & Hatano, 2008).

The current prevailing model of the relation between conceptual and procedural knowledge is the iterative model proposed by Rittle-Johnson and colleagues (2001) that assumes a bi-directional relation between the two types of knowledge and does not commit to the precedence of either conceptual or procedural knowledge; it rather assumes that either type of knowledge can trigger the learning process, depending on the child's prior experience with the domain in question. Importantly, this model predicts that increases in conceptual knowledge lead to increases in procedural knowledge and vice versa. In fact, supporting evidence for this model stems from experimental intervention studies that manipulate one type of knowledge and measure the effects on the other type (Rittle-Johnson et al., 2001).

The iterative model reconciles the contradictory findings in the field and accounts for many empirical findings, notably the well-documented finding that the two types of knowledge are typically highly correlated (Rittle-Johnson & Schneider, 2015). However, such correlations found at group level do not accurately reflect what happens at the individual level. Indeed, research shows that there are inter-individual differences in the way students develop the two kinds of knowledge putting a challenge to the iterative model (e.g., Canobi, 2004; Hallett, Nunes & Bryant, 2010; Hallett, Nunes, Bryant & Thrope, 2012). In particular, Hallett and colleagues

(2010, 2012) were the first to systematically study such individual differences in the area of fraction learning. They assessed conceptual and procedural knowledge of students at Grade 4 and 5 (2010) as well as at Grade 6 and 8 (2012) and they identified groups of students who were either strong or weak in both types of knowledge. However, they also consistently traced two substantial groups of students for whom there was discrepancy between the two types of knowledge (i.e., students in one group exhibited stronger procedural fraction knowledge, compared to their conceptual knowledge, and vice versa for students in the other group).

Before more progress can be made in understanding the relations between conceptual and procedural knowledge, more attention should be paid to the validity of measures of conceptual and procedural knowledge (Rittle-Johnson & Schneider, 2015).

### *Measures of conceptual and procedural knowledge*

Measuring conceptual and procedural knowledge is a challenging endeavor, particularly with respect to conceptual knowledge, which is considered a multi-dimensional construct. Indeed, procedural knowledge as knowledge of procedures is typically tested by asking students to carry out common school-taught algorithms. In contrast, assessing conceptual knowledge requires a great variety of tasks, as is evident considering, for example, the various components of conceptual knowledge of fractions listed above. Moreover, there is a great variety of task types used in this research area, even for the same mathematical idea (Crooks & Alibali, 2004). Rittle-Johnson and Schneider (2015) outlined two broad categories of tasks taken to provide explicit and implicit measures of conceptual knowledge. The first category (explicit measures) comprises tasks that urge the participants to articulate their understandings in a domain. Such tasks require, for example, explaining one's of answers in mathematical tasks, or explaining how procedures work, invoking the underlying principles. The second category (implicit measures) includes tasks requiring students to deal with mathematical tasks assumed to target conceptual knowledge such as representing or comparing fractions.

Using any of these types of tasks is challenging. For example, tasks assessing conceptual knowledge implicitly should not be familiar to the participants; else they might use a learnt procedure. Consider, for example, that the ability to represent fractions is deemed an indicator of conceptual knowledge. However, a task that requires representing a fraction using the area model (e.g., "shade the circle to represent 3/8") is so over-practiced in school, that it might actually be procedurized: a) Look at the number below the fraction bar; b) partition the circle in 8 equal parts; c) look at the number above the fraction bar; d) shade 3 parts.

Another, albeit related, problem is that tasks deemed "conceptual" may be accomplished by participants through procedural strategies. For example, when participants compare fractions mentally, it is typically assumed that they rely on their understanding of fraction magnitudes; restricting the use of paper and pencil, however, does not prevent participants who are familiar with the "cross-multiplication" strategy from accomplishing the task using a procedure (Faulkenberry, 2013). Such issues make interpreting results from questionnaire studies difficult.

On the other hand, explicit measures of conceptual knowledge require more than conceptual knowledge; they also require the ability to express effectively one's ideas. Such tasks in written questionnaires, in particular, may not be responded by the participants, more so if there are many of them.

As is evident from this discussion, measuring conceptual knowledge is a challenging endeavor. It should also be noted that measuring procedural knowledge as knowledge of procedures appears simpler only because it is delimited by the specific definition.

***Tapping on Greek secondary students' conceptual and procedural knowledge of fractions***
As expected based on the discussion above, the development of research tasks to measure conceptual and procedural knowledge has been an important part of the fours studies outlined in this article. To develop the tasks we were guided by the analysis of the components of conceptual knowledge presented above as knowledge of fraction representations; fraction features and properties; mathematical relations (equivalence, order); relations of fractions to other mathematical entities, in particular to natural numbers; the meaning of fraction operations and the principles to which they obey; and knowledge of the representational meaning of fractions. This does not mean that each assumed component corresponded to one or more tasks; rather, more than one of the components could be embedded in a task.

We were informed by prior studies on conceptual and procedural knowledge of fractions (e.g., Baroody & Hume, 1991; Hallett et al., 2010; 2012); on the development of instruments regarding rational number understanding (e.g., Van Hoof, Verschaffel & Van Dooren, 2015), on students' conceptual difficulties with fractions (e.g., Moss, 2005), in particular the inappropriate transfer of natural number knowledge to natural number tasks (e.g., Vamvakoussi & Vosniadou, 2010). We also considered literature on number sense (e.g., Reys & Yang, 1998).

Table 1 presents 9 main task categories that we used in all four studies, describing the corresponding tasks and referring to the assumed components of conceptual knowledge. Examples of items for each task categories are presented in Appendix A. One will notice that some items are presented in open-ended format, while others in multiple-choice format. We note that in Study 1 and 2 we used open-ended items. In Studies 3 and 4 similar items were used, albeit in multiple-choice format. The alternatives were informed by students' responses in Studies 1 and 2. Figure 1 presents an item (the "pie problem") in open-ended and multiple-choice format.

**TABLE 1**
*Examples of task types*

| Task Category | Task Description | Conceptual knowledge component(s) |
|---|---|---|
| A | Represent/evaluate a representation of a fraction using the area model | Representation |
| B | Represent/evaluate a representation of a fraction on the number line | Representation |
| C | Represent the result of a fraction operation (e.g., a product of fractions) | Representation, meaning of operations |
| D | Estimate the result of a fraction operation | Meaning of operations, fraction magnitude |
| E | Choose the appropriate operation to satisfy a constraint. | Meaning of operations, relation to natural numbers |
| F | Compare fractions mentally | Relations, fraction magnitudes |
| G | Mind a difference between fractions and natural numbers | Properties, relations to natural numbers |
| H | Consider the unit of reference in contextualized problem | Representational meaning |
| J | Use an appropriate operation to model a situation | Representational meaning, meaning of operations |

**FIGURE 1**
**The pie problem**

**Open-ended item**                                          **Multiple-choice item**

Maria says: I ate $\frac{3}{5}$ of my pie.                    Somebody ate $\frac{3}{5}$ of a pie. This person ate:

Kostas says: So you ate 3 pieces of pie.                     **a.** 3 pieces of pie

➤ Do you agree with Kostas?                                  **b.** 3:5 pieces of pie
  Explain your answer.                                        **c.** None of the previous responses

*Example of a research item in open-ended and multiple-choice format*


**OVERVIEW OF THE STUDIES**

***Study 1: General trends in Greek secondary students' conceptual and procedural fraction knowledge***

Study 1 (Bempeni & Vamvakoussi, 2014) was a cross-sectional study that investigated the development of conceptual knowledge of fractions from grade 7 to grade 9. We designed a questionnaire consisting of 24 open-ended items targeting conceptual knowledge of fractions along the lines described above. The difficulty of items varied, from very simple ones (e.g., representing a proper fraction using the area model) to quite challenging ones (i.e., items targeting the dense ordering of fractions). In addition, the questionnaire included 4 procedural items on standard school-taught procedures, namely on fraction operations (e.g., "compute the sum 10/63+8/9"). We administered this questionnaire to 80 7th and 9th graders. The students were asked to solve the conceptual tasks and explain their solutions; and to carry out the fraction operations.

The results showed no significant difference between the two age groups' overall performance, and no significant difference in the overall performance for conceptual tasks, nor for procedural tasks. However, we found that older students relied significantly more on procedural strategies (i.e., school-taught procedures). Looking at each task separately, we found that 9th graders performed significantly better in 3 conceptual tasks, albeit relying on school-taught transformation strategies. For example, they converted fractions into decimals in order to place them on the number line; and they converted 3/2 and 13/27 into similar fractions in order to compare them.

Performance in procedural tasks was fairly good (at least 70% succeeded in each task), whereas performance in conceptual tasks varied widely (as expected). An interesting pattern emerged: Although the great majority of students were apt to carry out the four fraction operations, about 60% failed in tasks targeting conceptual understanding of the operations (e.g., in estimating the sum 12/13+7/8). In addition, although practically all students were able to construct an area model for a proper fraction, about 25% asserted that the shaded part of the triangle in item A1 (Appendix A) was "one-third"; and more than half of the students failed to represent an improper fraction (5/3), or explicitly stated that "it is not possible to choose five parts out of three". Further, about half asserted that "eating 3/5 of a pie" necessarily means "eating 3 pieces of a pie" (see Figure 1). Finally, we also traced some students who were flawless in the procedural tasks, but failed in even the simplest conceptual tasks (e.g., item A1 in

Appendix A), indicating an asymmetry between the two types of knowledge. We focused on this issue in Study 2.

### Study 2: Inter-individual differences in conceptual and procedural knowledge of fractions

In Study 2 (Bempeni & Vamvakoussi, 2015) we recruited 7 9th graders to participate in in-depth task-based interviews. The selection of the participants was not random. First, based on their school grades, all participants could be characterized as medium to high level students in mathematics. Second, they shared the same mathematics tutor who provided information on their mathematical competence. Based on this information, we had reasons to expect some variation in their conceptual and procedural knowledge of fractions.

We used an instrument with 30 fraction open-ended tasks, adjusting and extending the collection of conceptual tasks used in Study 1, and adding few procedural tasks (e.g., operations with mixed numbers). We also explicitly asked students not to apply school-taught procedures in certain fraction comparison tasks that could be tackled with conceptual strategies (e.g., one proper and one improper fraction; similar to item F in Appendix A, albeit in open-ended format). The participants were asked to solve the tasks thinking aloud and explaining their answers.

Based on the analysis of their responses in terms of accuracy and strategy used (procedural/conceptual), the participants were distributed in three profiles. The *Conceptual-Procedural* profile consisted of three students who showed robust conceptual knowledge of fractions, combined with procedural knowledge.

The *Procedural* profile consisted of three students who succeeded in practically all procedural tasks, but failed systematically in conceptual tasks. When restricted from using school-taught procedures, they exhibited severe lack of fraction understanding. For example, they asserted that a fraction is bigger, when it has larger terms; that multiplication always results in larger results; and, similarly to several students in Study 1, they claimed that eating 3/5 of a pie necessarily means eating 3 pieces out of 5 pieces of pie, because "the denominator shows how many pieces there are, and the nominator shows how many pieces you take".

Finally, the *Conceptual* profile was represented by one student that failed in all tasks requiring procedural knowledge, but managed to deal successfully with most of the conceptual tasks. This student excelled in tasks targeting fraction representations; exhibited a variety of conceptual strategies; and provided sophisticated explanations in many conceptual tasks.

The findings of Study 2 were consistent with the findings of Hallett and colleagues (2010, 2012), indicating that that there are individual differences in the way students develop the two types of knowledge. Moreover, they illustrated the possibility that these differences can be extreme, even at grade 9, that is, for older students that the participants in the studies of Hallett and colleagues. In following studies, we attempted to further investigate this issue, shifting from qualitative to quantitative methods.

### Study 3: Developing a valid and reliable instrument

In Study 3, we developed and evaluated a new instrument measuring conceptual and procedural knowledge of fractions (Bempeni, Poulopoulou, Tsiplaki & Vamvakoussi, 2018). In its initial form, the instrument consisted of 39 items, 12 procedural items and 27 conceptual items. The procedural items were typical school tasks testing for knowledge of taught procedures. For example, students were asked to carry out fraction operations and operations with mixed numbers; to find an equivalent fraction; to simplify complex fractions; and to compare dissimilar fractions using the standard procedure.

The conceptual tasks were based on our materials from Study 1 and Study 2 which were adjusted and enriched when necessary, based on our experience with testing these tasks with students. In this study we opted for multiple-choice items, instead of open ones (see also Van Hoof et al., 2015), with a view to discourage the use of procedural strategies in conceptual tasks (e.g., in comparison tasks), an issue that, as discussed, is highlighted in the literature (e.g., Faulkenberry, 2013) and was noticeable in Study 1.

We conducted a clinical pilot study with 61 students and asked 6 mathematics education experts to assist in the evaluation of the instrument. The instrument was assessed with respect to a) face validity and content validity, through the experts' feedback on clarity, accuracy, and relevance of the instrument; b) convergent and divergent validity, via multitrait analysis; c) internal consistency, calculating Cronbach's alpha; and d) external consistency with the test-retest method, calculating the intra-class correlation coefficient.

The instrument showed strong face validity given that all items were assessed as clear and accurate by the experts, who were also highly consistent with each other in rating the relevance of each item to the aim of the instrument (Content Validity Index $=1>.83$). All items of the procedural scale showed convergent validity and divergent validity by demonstrating high correlation with the procedural scale and low correlation with the conceptual scale, respectively. However, eight items of the conceptual scale showed low correlation with the conceptual scale or higher than expected correlation with the procedural scale. Further, the value of intra-class correlation coefficient was high (above 0.8) for all procedural items, but below 0.5 for five conceptual items.

The problematic items were removed, resulting in an instrument consisting of 14 conceptual and 12 procedural tasks, with good indicators of validity, reliability, and objectivity, which we used in Study 4. However, removing conceptual items from the questionnaire is not an insignificant matter, and we will come back to this issue in the discussion.

### Study 4: Testing for inter-individual differences in a quantitative study

In Study 4 (Bempeni et al., 2018) we administered the aforementioned instrument to 126 ninth graders and we analyzed the data using cluster analysis, testing for individual differences. Following Hallett and colleagues (2010, 2012), we used in the analysis the residualized scores of the procedural and the conceptual scales, the raw scores being the percentages of correct answers out of the total of answered questions. This is because the two scales are expected to be correlated (Rittle-Johnson & Schneider, 2015), and this method provides a way to exclude the common part of variation from both scales. It should be noted that these scores do not represent absolute magnitudes of each type of knowledge; rather, they indicate discrepancies between the two types of knowledge. For example, a positive residual with respect to procedural knowledge means that a person's procedural knowledge is stronger than expected given their conceptual knowledge, and vice versa for negative residuals.

The cluster analysis used the k-means method and Euclidean distance as a distance measure and the optimal number of clusters was determined via statistical methods to be four. Two of these clusters comprised students who did not show great discrepancies between the two types of knowledge and their performance was either equally high in both type of tasks (N=22, 17.5%) or equally low in both types of tasks (N=31, 24.6%). On the contrary, the two remaining clusters comprised students who either performed better than expected in conceptual tasks given their performance in procedural tasks (Conceptual Profile, N=21, 16.7%); or performed better in procedural tasks given their performance in conceptual tasks (Procedural Profile, N=52, 41.3%).

These results supported the hypothesis that there are individual differences in the way that students develop the two type of knowledge and were consistent with the findings of our previous studies, and also with the results of Hallett et al. (2010, 2012). Moreover, our findings provided evidence that these differences may persist and remain salient for older students.

We note that within the Procedural Profile, we traced few extreme cases of students who excelled in the procedural items, but showed a severe lack of conceptual understanding, similar to students in Study 2. For example, one student achieved 100% score in the procedural tasks but only 14.29% in conceptual tasks, failing even in some of the simplest ones. We did not trace such extreme cases within the Conceptual Profile. Nevertheless, there were students with extremely low performance in procedural tasks, who still managed to deal with several of the conceptual tasks. For example, a student in the Conceptual Profile failed in all procedural tasks and responded correctly to 50% of the conceptual tasks. Interestingly, this particular student was able to estimate a sum of fractions (i.e., item G in Appendix A) despite the fact that she had failed to perform fraction addition.

## CONCLUSIONS-DISCUSSION

In this article we presented an overview of four studies investigating secondary students' conceptual and procedural knowledge of fractions, qualitatively as well as quantitatively. The findings of these studies converge on the conclusion that there are inter-individual differences in the development of these types of knowledge (see also Hallett et al., 2010, 2012). In addition, they indicate that such differences may remain salient up to ninth grade; and that they may be even extreme, in the sense that one type of knowledge may be far more developed than the other (Study 2, Study 4). Our findings are strengthened by the fact that we measured conceptual and procedural fraction knowledge reliably and validly via a new instrument (Study 4).

### Theoretical issues
Given the empirical evidence regarding inter-individual differences in the relative strength of conceptual and procedural knowledge, the question arises: Do such findings put a challenge to the iterative model proposed by Rittle-Johnson and colleagues (2001)? Rittle-Johnson and Schneider (2015; see also Rittle-Johnson, Schneider & Star, 2015) acknowledge the presence of inter-individual differences. They argue that taking an iterative view on the development of conceptual and procedural knowledge does not imply the assumption that "the two types of knowledge are equally developed at any given time" (Rittle-Johnson et al., 2015, p.590). However, considering the age and education level of our participants together with the fact that they have been exposed to fraction instruction since Grade 3, one would have expected a more symmetrical relation between the two types of knowledge. In particular, it can be questioned whether extreme asymmetries such as the ones exhibited, for example, by students of Study 2, can be explained by the iterative model.

We believe that these findings highlight the fact that increases in one type of knowledge do not *necessarily* lead to substantial increases in the other type of knowledge. Rather, there are (at least) two factors than need to be taken into consideration. First, given that this is case of instruction-induced knowledge development, the quality of instruction matters, particularly with respect to procedural knowledge. Rittle-Johnson and colleagues (2015) touched on this point, when they reviewed studies showing that improvements in procedural knowledge can lead to improvement in conceptual knowledge. The practice problems used in such studies were

carefully constructed and sequenced based on conceptual principles. Thus, it is not the case that any type of teaching for procedural knowledge triggers the development of conceptual knowledge.

Second, the intention of the learner must also be taken into consideration. Indeed, it arguably takes engagement, effort, and reflection to extract abstract principles from repeated experiences with practice problems, even if these are carefully chosen. Similarly, conceptual knowledge manifested, for example, as the ability to estimate fraction magnitudes does not necessarily lead to procedural skill; rather it takes some extra effort to learn and become able to carry out the standard procedure for fraction comparison accurately. Whether this effort is invested or not depends on the learner's intention.

*Methodological issues*

Developing a valid and reliable instrument to measure conceptual and procedural fraction knowledge (Study 3) has been a challenging endeavor. In fact, there were several theoretical and methodological issues that we had to tackle in the process, and some difficult decisions to make. First, measures of procedural knowledge were easy to construct and validate. However, this is due to the fact that we adopted a simple, perhaps over-simplified, definition of procedural knowledge (Star, 2005), as discussed in the introduction.

Second, measures of conceptual knowledge were particularly challenging. On the one hand, it is important to address various aspects of conceptual knowledge (Rittle-Johnson & Schneider, 2015), which requires the use of a great variety of tasks. On the other hand, this variety makes it difficult to construct a measure with good indicators of convergent validity, resulting to the exclusion of tasks (Study 3). Further, the use of multiple-choice items does not exclude the possibility that students in fact use procedural strategies when dealing with conceptual tasks, which was observable in Studies 1 and 2, but not in Studies 3 and 4. This might be an explanation for the fact that some conceptual tasks showed higher correlation with the procedural scale than expected (Study 3), and they also needed to be excluded. For example, locating fractions on the number line were among the problematic conceptual tasks in Study 4, presumably because the students, similarly to the students in Study 1, used transformation strategies to deal with the task. Excluding items may result in a more robust instrument; however a lot of useful insights in students' understandings are lost. Consider, for example, the "pie problem" (Figure 1). This task targets a fundamental, aspect of fraction understanding, but we had to remove it from the instrument (Study 3), because the great majority of students, even some of the ones with overall good performance, failed. Thus, this task was not useful in discriminating between "conceptual" and "procedural" students, but removing it meant ignoring an important aspect of conceptual knowledge about fractions that students appeared to lack.

*Few remarks on instruction*

The latter remark brings us to a consistent finding across all four studies: Students failed in elementary, yet fundamental conceptual tasks. Consider again the "pie problem" (Figure 1) which was used in all studies. This task was challenging for half the students in Study 1; for three out of seven middle-to-high level students (based on their school grades) in Study 2; and for the great majority of students who participated in Study 3.

This raises certain issues for instruction. First, Study 1 indicates that conceptual knowledge does not improve from $7^{th}$ to $9^{th}$ grade, despite the fact that during these years Greek students recapitulate content regarding fractions and are introduced to rational numbers. However, what seems to change is that students tend to rely more on procedural strategies. This

may conceal their lack of conceptual fraction understanding is some cases (e.g., when placing fractions of the number line), but not in tasks that cannot be tackled with school-taught procedures (e.g., the pie problem). Second, it appears that students with very poor understanding of fractions still manage to get good grades at school (Study 2). Third, the majority of the students in Study 4 were placed in the *Procedural* profile (i.e., they did better in the procedural tasks, than one would expect based on their performance in the conceptual tasks). These are quite strong indications that students' school experiences are more favorable to the development of procedural knowledge. In other words, instruction appears to still over-emphasize procedural knowledge, neglecting students' conceptual difficulties with fractions (Moss & Case, 1999). This assumption cannot, however, explain the existence of (fewer) students who have stronger conceptual, than procedural knowledge.

### *Further research*

The source of individual differences in conceptual and procedural knowledge is still an open question, despite the fact that several hypotheses have been formulated and tested. Factors such as prior knowledge in the domain (Schneider, Rittle-Johnson & Star, 2011); cognitive profile (Gilmore & Bryant, 2008; Hallett et al., 2012); general conceptual of procedural ability as well as school experience measured as school attendance have been tested (Hallett et al., 2012), with unsatisfactory results.

We believe that the learner's intention when engaging with mathematics is an important factor worth investigating in this respect. For example, investigating students' learning approaches to mathematics, together with their conceptual and procedural knowledge of fractions, Bempeni and Vamvakoussi (2015) found that students on the conceptual side had different study goals compared to students on the procedural side (understanding vs. school achievement, respectively). Further theorizing and research is needed in this direction.

## ACKNOWLEDGEMENT

## REFERENCES

Baroody, A. J., & Hume, J. (1991). Meaningful mathematics instruction: The case of fractions. *Remedial & Special Education*, *12*(3), 54-68.

Bempeni M., & Vamvakoussi X. (2014). Seventh and ninth graders' conceptual and procedural knowledge of fractions: what changes with school experience? *Paper presented at the 9th International Conference on Conceptual Change,* Bologna, Italy.

Bempeni, M. & Vamvakoussi, X. (2015). Individual differences in students' knowing and learning about fractions: Evidence from an in-depth qualitative study. *Frontline Learning Research, 3,* 17-34.

Bempeni, M., Poulopoulou S., Tsiplaki I., & Vamvakoussi X. (2018). Individual differences in fractions' conceptual and procedural knowledge: What about older students? In E. Bergqvist, M. Österholm, C. Granberg & L. Sumpter (Eds.), *Proceedings of the 42ⁿᵈ Conference of the International Group for the Psychology of Mathematics Education* (pp. 147-154). Umeå, Sweden: PME.

Canobi, K. H. (2004). Individual differences in children's addition and subtraction knowledge. *Cognitive Development, 19,* 81-93.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review*, *34*(4), 344-377.

De Corte, E. (2004). Mainstreams and perspectives in research on learning (mathematics) from instruction. *Applied Psychology: An International Review, 53*(2), 279-310.

Faulkenberry, T. J. (2013). The conceptual/procedural distinction belongs to strategies, not tasks: A comment on Gabriel et al. (2013). *Frontiers in Psychology, 4*, Article 820.

Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition*, *13*(3), 343-359.

Gilmore, C. K., & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. *British Journal of Developmental Psychology, 26*, 301-316.

Hiebert, J. (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.

Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology, 102*, 395-406.

Hallett, D., Nunes, T., Bryant, P., & Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: The role of abilities and school experience. *Journal of Experimental Child Psychology, 113,* 469-486.

Inagaki, K., & Hatano, G. (2008). Conceptual change in naive biology. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 240–262). New York: Routledge.

Kilpatrick, J., Swafford, J. O., & Findell, B. (2001). *Adding it up: Helping children learn Mathematics*. Washington, DC: National Academy Press.

Moss, J. (2005). Pipes, tubs, and beakers: New approaches to teaching the rational-number system. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: History, math, and science in the classroom* (pp. 309-349). Washington DC: National Academies Press.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum, *Journal for Research in Mathematics Education, 30*, 122-147.

Nunes, T., & Bryant, P. E. (2015). The development of mathematical reasoning. In L. S. Liben, U. Müller, R. M. Lerner, & K. E. Adolph (Eds.), *Handbook of Child Psychology and Developmental Science (vol. 2): Cognitive Processes* (7th ed., pp.715-762). Hoboken, New Jersey: Wiley.

Reys, R. E., & Yang, D. C. (1998). Relationship between computational performance and number sense among sixth and eighth grade students in Taiwan. *Journal for Research in Mathematics Education, 29,* 225-237.

Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In R. Kadosh & A. Dowker (Eds.), *Oxford Handbook of Numerical Cognition* (pp. 1118-1134). Oxford: Oxford Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346-362.

Schneider, M., Rittle-Johnson, B, & Star, J. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Journal of Developmental Psychology, 47*, 1525-1538.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, *127*(4), 377-397.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36,* 404-411.

Star, J. R., & Stylianides, G. J. S. (2013). Procedural and conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics, and Technology Education, 13*(2), 169-181.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181-209.

Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics, 89,* 39-56.

## APPENDIX A
Examples of research tasks

**A1.** Is the shaded part of the triangle $\frac{1}{3}$ of its area?
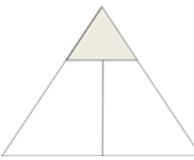
**A2.** The gray bar is $\frac{3}{2}$ X, where X is one of the white bars. Which one is X?
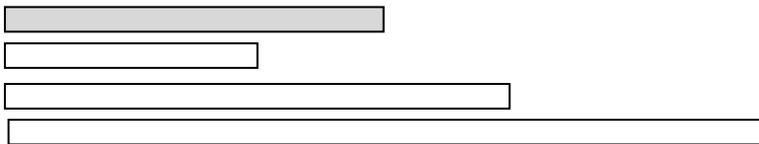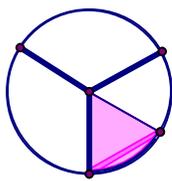
    **a.** The first white bar      **b.** The second white bar      **c.** The third white bar
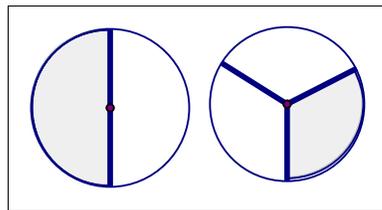
**B.** Which fraction corresponds to point A on the number line?

**C.** Which of the following represents $\frac{1}{2}$ of $\frac{1}{3}$ ?

    **a.**            **b.**          **c.** None of the previous

**D.** Without computing, can you tell if the sum $\frac{12}{13} + \frac{7}{8}$ -is

**a.** greater than $\frac{1}{2}$      **b.** smaller than $\frac{1}{2}$      **c.** I cannot tell if I do not carry out the addition

**E.** Without computing, can you choose the operation that makes this inequality true?

$$121 \ \square \ \frac{3}{4} > 121$$

    **a.** Multiplication      **b.** Division      **c.** I cannot choose without computing

**F.** Without computing, can you tell which of $\dfrac{211}{423}$ and $\dfrac{31}{29}$ is greater?

**a.** $\dfrac{211}{423}$ is greater than $\dfrac{31}{29}$  **b.** $\dfrac{31}{29}$ is greater than $\dfrac{211}{423}$  **c.** I cannot tell if I do not make them similar

**G.** Are there any numbers between $\dfrac{2}{5}$ and $\dfrac{3}{5}$? If yes, how many?

**H.** Maria bought one pizza from Lucullus and ate one quarter of it. John bought one pizza from Vesuvius and ate half of it. Could you tell who ate more pizza?

**I.** Eleni has two drawers full with socks. In the first drawer, $\dfrac{1}{2}$ of the socks are white. In the second drawer, $\dfrac{1}{3}$ of the socks are white. Could you tell how many white socks does Eleni have?