# Recognizing pattern rules in an early childhood classroom: the role of manipulatives

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**ABSTRACT**
*This study explores the way that the pattern rule is developed in preschool age and the contribution of the manipulatives to this recognition. For this purpose, 23 preschoolers were tested in pre- and post-tests which were designed to examine the development of the pattern rule. From the results it appears that preschoolers recognize and generalize the rule in 7 steps and that the manipulatives help children on the one hand, in this recognition and generalization and, on the other hand, to construct their own manipulatives, which they then use in order to deal with other relevant mathematical situations.*

**KEYWORDS**
*Pattern, rule, manipulatives, generalization, preschool*

**RÉSUMÉ**
*Cette étude explore la façon dont la règle du modèle est développée à l'âge préscolaire et la contribution des manipulateurs à cette reconnaissance. À cette fin, 23 enfants d'âge préscolaire ont été testés lors de pré et post tests conçus pour examiner le développement de la règle du modèle. D'après les résultats, il apparaît que les enfants d'âge préscolaire reconnaissent et généralisent la règle en 7 étapes et que les manipulateurs aident les enfants d'une part, dans cette reconnaissance et généralisation et, d'autre part, à construire leurs propres manipulateurs, qu'ils utilisent ensuite pour traiter avec d'autres situations mathématiques pertinentes.*

**MOTS-CLÉS**
*Modèle, règle, matériel, généralisation, préscolaire*

## INTRODUCTION

Recently, curricula changes regarding mathematics have focused on the development of the mathematical structure in early algebraic thinking, as it is considered important for the abstraction and generalization of mathematical ideas. As a result, patterns and algebra have been included in early childhood education (Department for Education and Skills, 2001; National Council of Teachers of Mathematics, 2000), while the research in this field showed that the development of algebraic reasoning can be developed by students in primary education (Mulligan et al., 2006; Schliemann et al., 2003;). However, admitting that teaching Mathematics in early years is not an easy endeavor, researchers highlight the importance of manipulatives and correlate their use with students' performance (Raphael & Wahlstrom, 1989), while they emphasize that their use requires careful design and anticipation (Szendrei, 1996). On the other hand, mathematical conceptualization in order to be accomplished requires generalization, (Zazkis, Liljedahl, & Chernoff, 2008), which however complicates students (Becker & Rivera, 2005). Although there is some evidence concerning young learners' endeavors to generalize pattern rules (Warren, 2004), generalization has not been studied systematically in early childhood.

Thus, this study sought to investigate the way preschoolers conceptualize and generalize pattern rules, as well as the importance of the use of manipulatives to this conceptualization.

## THEORETICAL BACKGROUND

The development of algebraic thinking in early mathematics education, is mainly associated with the structure of the pattern, namely the recognition of the repeating composite unit which constitute the rule of the pattern (Tzekaki, 2010). The elements of the pattern are organized on the basis of the rule which indicates the specific relationships among them.

Concerning the development of structure in early childhood, Mulligan, Mitchelmore, & Prescott (2005) report four development stages: *pro-structural, emergent, partial and structural*, based on whether the children focused on the structural features of the pattern or its superficial features. Regarding the identification of a repeating unit in repeating and spatial patterns, researchers stress that the teaching intervention which promotes patterning concepts, helps children to identify the repetition of the unit and extend this knowledge to other kind of patterns (Papic & Mulligan, 2007). It is also, highlighted that structure recognition can be developed from preschool age (Papadopoulou, 2012) as children can recognize relationships through teaching interventions which support more general and coherent forms of thought (Blanton & Kaput, 2004).

As it is mentioned earlier mathematical conceptualization requires generalization, as there is no development of mathematical concepts without it, issue that has been and still is of concern to the mathematical scientific community. So, in order to conceptualize the rule of a pattern, children have to generalize the rule. Regarding generalization, Kaput (1999) argues that in algebraic generalization the focus is on the patterns, processes, structures and relationships through and among them and Radford (2006) claims that in algebraic generalization a local feature is generalized to all terms of the sequence. Carraher, Martinez and Schliemann (2008) report that in mathematical generalization some properties or techniques are applied to a large set of mathematical objects or situations.

We notice that the above researchers have two common points: a) during generalization, the recognition of a common element/feature/property or technique, which is repeated in a mathematical situation, takes place (Carraher et al., 2008; Kaput, 1999; Radford, 2006) and (b) during generalization the focus moves away from local and passes to general (Carraher et al., 2008; Kaput, 1999; Radford, 2006).We also know that children gradually shift from the specific content of the mathematical situation that they are called to face in a mathematical situation, to the invariable features of the concept, namely its generalization (Lannin, Townsend, & Barker, 2006).

To create mathematical environments that help the child to understand more deeply the mathematical meanings, the teaching practices and the use of manipulatives play an important role play. With regard to the teaching practices, the social interaction through exchange of ideas, solutions, discussions, explanations, documentations (Radford, 2006), the development of reflection and error recognition (Pappas, Ginsburg, & Jiang, 2003), and the verbal formulation of thoughts, which helps to the development of the concepts as it mediates to their formulation (Vergnaud, 1996), are really helpful for the development of concepts. In accordance with the above and in an effort to develop children's understanding of mathematical concepts, the role of the teacher is to help young learners express their conclusions (Tzekaki, 2010).

Regarding manipulatives, they are used in teaching and learning processes, in order to assist in the understanding of mathematical concepts, and in the construction and maintenance of new mathematical ideas. Manipulatives are closely linked to the learning process (Braswell et al., 2011) and their use in educational processes is considered useful. They facilitate the construction of mathematical concepts by children during their interaction and the exchange of their ideas (Cobb, 1991). It is also argued that manipulatives make mathematical concepts more specific (McCulloch Vinson, 2001) and allow the child to form various images in his/her mind, and elaborate mentally abstract concepts (Moyer, 2001). The reason and the way that manipulatives are used in math teaching is also linked to the teachers' beliefs about how students learn mathematics (Moyer, 2001). There are teachers who use manipulatives in order to reform the teaching of mathematics, without understanding the significance of representations and their role in the learning process (Grant, Peterson & Shojgreen-Downer, 1996).

Researchers emphasize the importance of the context in which mathematical manipulatives are used, namely the way students work with manipulatives, the purpose they work for, and the way they communicate and interact (Ball, 1992). The ability of children to engage in a variety of mathematical experiences through the use of manipulatives and the justification of their actions through reflection is also important (Clements & McMillen, 1996). Also, it is emphasized that arguments which students develop in order to document their views are supported by manipulatives (Driver, Newton, & Osbone, 2000).

In the light of the above, this research aims to investigate the degree that preschoolers are able to generalize the rule (structure) of repeating and growing patterns and the role of manipulatives to the development of this generalization. Can preschoolers conceive the rule of the pattern? How is the pattern rule developed by preschoolers? How do manipulatives support the understanding of the rule? How do students use this material? In this article, an attempt will be made to answer these questions, and the research that will be presented is part of a broader one, which has been carried out in three more fields of mathematics.

In the present study the *generalization of a pattern* is defined as the *recognition of a repeating rule and the relationships of its elements*. *Generalization at local level* concerns the recognition of the repeating rule and the relationships of its elements in a particular pattern that the child is confronting, while *generalization at general level* concerns a general recognition of the patterns beyond the perceptual field, which means that there is a repeating rule and that the elements of the rule are interrelated with specific relationships.

## METHODOLOGICAL FRAMEWORK

### *Participants and procedure*

This paper reports on a small-scale exploratory study, which investigated how young children generalize the rule of the repeating and growing patterns. 23 preschoolers aged 5-6 years old participated, attending a public school in eastern Thessaloniki. The children were never engaged in formal learning activities about patterns up to that moment. This research was aimed at preschool-aged children at typical classes of kindergarten and for this reason the children were not selected on the basis of some specific characteristics.

The research tools were pre- and post-tests, with 7 tasks, the same for both type of tests, which concern children's works and had been implemented in a 3 weeks span. The pre and post tests were structured on the basis of generalization analysis of the pattern rule and were given to each child individually. Firstly, each child had to face a task in repeating patterns and in a growing pattern and later answer questions to an interview of approximately 20 minutes. The interview questions were designed in order to examine the understanding of the rule of the pattern. For example, the children were asked: "What did you think to find it? What did you notice?", "Show, which is the small piece that is repeated?", "Show, where does that small piece start and where does it end?" (Papadopoulou, 2017, p. 108)

A teaching intervention was implemented including 12 activities which were organized in 6 phases and the duration of each phase was 45-50 minutes. The field of patterns was analyzed in axes of mathematical situations, which analyze the concept of pattern in mathematical actions that allow children to shift from a morphological awareness to the recognition of the rule and the relationships of its elements. These axes are: reproducing and extending a given pattern, completion of a missing element in a given pattern, finding an error in a given pattern, coping a rule and transferring it to another material, extending a pattern and creating a pattern by the children themselves (Tzekaki, 2010).

The most characteristic part of this intervention are the teacher's questions at the closure of each activity that lead children: (a) to reflect on their own actions (Dreyfus, 2015), (b) to formulate some conclusions regarding these actions (Radford, 2006) and (c) to record these conclusions (Radford, 2006) in a semiotic way (by drawing). Initially, these questions address to a first *local level of generality* and relate to the recognition of the rule and the relationships of its elements, of the specific pattern that the child has in front of him/her, for example on the table. Some indicative *generalization questions at local level* are the following (Papadopoulou, 2017, p. 303): "Which is the small piece that is repeated?", "What is the beginning and the end of this small piece?", "If we were asked to draw this design, what should we remember to do?", "In this design, was there a row of the cubes' sizes or were these randomly put? What is that row?".

Then, the questions address to a second *general level of generality*, encouraging announcements of the rule, beyond the perceptual field. Indicative *generalization questions at general level* (Papadopoulou, 2017, p. 303): "So, what do I need to do in order to remember a design?", "Do I need to remember the whole design or can I do something else?".

The following parameters were taken into account for the design of the manipulatives, so as to support:

1. the classification of patterns in terms of structure, content, evolution, as well as overlaps and arithmetic patterns (Tzekaki, 2010),
2. the generalization as defined, that is to help the child identify, describe and verbally formulate the rule of a pattern; and
3. the teaching methodology chosen to develop generalization, and included discussions, documentation, error recognition, strategy development, the use of prior generalized knowledge, reflection, *systematic formulation and recording of local and general conclusions* by the children themselves, based on the *local and general level generalization questions*.

### *Data analysis*

In order to examine children's responses to the pre- and post-tests, a conceptual development model of the rule was created, which illustrates how the understanding of the rule and the relationships of its elements is developed from the real to the abstract world. The approaches that this model is based on concern older children than the preschoolers, but the basic idea of these approaches was derived in order to be applied in kindergarteners. This model is represented in table 1 that follows.

**TABLE 1**

*The 3-level conceptual development model*

| Levels | Content of levels |
|---|---|
| **Level 0** | No recognition of the rule and its repetition. |
| **Level 1** | Holistic approach of the rule: development of procedural understanding (Gray & Tall, 1994; Haapasalo & Kadijevich, 2000) of the rule and its relationships. |
| **Level 2** | Conceptual understanding of the rule: conceptual development (Grey & Tall, 1994; Haapasalo & Kadijevich, 2000) of the rule and its relationships, with verbal formulation (Vergnaud, 1996). |

## RESULTS

From the analysis of the children's responses on the basis of the 3-level conceptual development model, each level was expanded into stages that resulted from the classification of children's responses, based on whether they recognize and formulate the rule of the pattern and the type of this recognition (procedural/conceptual). The following table 2 briefly describes how the pattern rule is developed by preschoolers.

**TABLE 2**

*The 7 stages of the generalization of the rule*

<table>
<tr><th>Levels</th><th>Stages</th><th>Content of the stages</th></tr>
<tr><td>Level 1: no recognition</td><td>Stage 1</td><td>The child does not recognize any relationship between the elements of the pattern.<br>Typical response: s/he makes a painting that is not related to a pattern, e.g. s/he paints one person.</td></tr>
<tr><td rowspan="3">Level 2: holistic approach/ procedural understanding</td><td>Stage 2a</td><td>The child recognizes some relationships that do not structure the rule. Typical response: the child makes a drawing without a rule and to the question: "Which is the small piece that is repeated?", s/he states an element that appears, e.g. 2-3 times in her/his drawing, e.g. "The square".</td></tr>
<tr><td>Stage 2b</td><td>The child recognizes more relationships which also do not structure the rule.<br>Typical response: the child makes a drawing without a rule, but there are elements that appear 2-3 times and when asked if there is a small piece that is repeated, s/he mentions at least 2 of these elements, e.g. "The triangle and the square".</td></tr>
<tr><td>Stage 3</td><td>The child recognizes all the relationships: they are not relationships that structure the rule, but relationships that result from the correspondence of the elements one to one of the repeated rule.<br>Typical response: Though the child does the pattern right, when asked which small piece is repeated, s/he mentions the elements individually, corresponding them one to one, e.g. "It's the circle, the square, and the triangle, because here is the circle, the square and the triangle."</td></tr>
</table>

| | | |
|---|---|---|
| Level 3: conceptual understanding | Stage 4a | The child begins to recognize some of the relationships that structure the rule, *since s/he recognizes a series of elements in the pattern*.<br>Typical response: When asked which is the small piece that is repeated, s/he “reads” the whole design, e.g. "it is big-small-small-middle-big-small-small-middle-big-small-small-middle". |
| | Stage 4b | The child recognizes the relationships that construct the rule, but these relationships are not stable.<br>Typical response: s/he formulates the rule correctly, but makes a mistake in a repetition. |
| | Stage 5 | The child recognizes the relationships that construct the rule and formulates it verbally.<br>Typical response: s/he says the rule correctly and shows all repetitions correctly with consistency. |

The following table 3 summarizes the number of children at each stage in the pre- and post-tests, in all tests in the patterns.

**TABLE 3**

*The number of children in the 7 stages of the development of the rule*

| **Stages** | **Tests** | **Reproducing** | **Completion/ geometric** | **Completion/ size** | **Finding an error** | **Copy of the rule** | **Extending** | **Creation** |
|---|---|---|---|---|---|---|---|---|
| Stage 1: no recognition of the rule and its relationships | Pre | 19 | 4 | 2 | 5 | 9 | 6 | 8 |
| | Post | 1 | 0 | 0 | 1 | 0 | 2 | 0 |
| Stage 2a: recognition of some relationships which are not related to the rule | Pre | 3 | 5 | 5 | 8 | 6 | 1 | 2 |
| | Post | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Stage 2b: recognition of more relationships which are not related to the rule | Pre | 0 | 3 | 6 | 9 | 6 | 0 | 8 |
| | Post | 1 | 2 | 0 | 6 | 0 | 0 | 0 |
| Stage 3: recognition of all relationships with corresponding, which are not related to the rule | Pre | 0 | 9 | 8 | 1 | 2 | 9 | 2 |
| | Post | 1 | 0 | 0 | 0 | 0 | 4 | 0 |
| Stage 4a: recognition of some relationships of the rule | Pre | 0 | 2 | 2 | 0 | 0 | 5 | 3 |
| | Post | 4 | 0 | 8 | 2 | 9 | 4 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Stage 4b: recognition of more relationships of the rule | Pre | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| | Post | 14 | 6 | 7 | 4 | 5 | 12 | 0 |
| Stage 5: recognition of all relationships of the rule | Pre | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Post | 2 | 14 | 8 | 10 | 9 | 1 | 23 |

Findings show that before the teaching intervention, the children do not recognize any relationship among the elements of the pattern (stage 1) or have a holistic and procedural perception of the rule (stages 2a, 2b,3), which means that they do not recognize that there is a repeated rule and that its elements are linked with specific relationships. However, they correspond the elements of the pattern one to one, which means that they see how the elements of the pattern are placed in the first repetition and they correspond these elements one to one in the next repetitions.

After the teaching intervention, the children gradually recognized the repeated rule and the relationships among its elements (stages 4a, 4b, 5), they developed the rule awareness and arrived at relevant conclusions.

*The supportive role of manipulatives to the rule recognition*

The examination of the teaching intervention provided some information about the way manipulatives supported the understanding of the pattern rule. The use of concrete manipulatives seems that helped children to construct the rule. For example, when children had to create their own pattern by drawing, some children borrowed the idea from a pattern that was previously given to them, in which they coined their own rule and made their own pattern. For example, from a pattern with ABAB structure (A = large circle, B = a little circle) which was given to the children in an activity, a girl constructed a pattern with ABBB structure (A circle large, B = circle small) explaining: “Yes, I did as with the black balls, eh! not exactly the same, in another order, look! Big ball, small ball, small again and small again”.

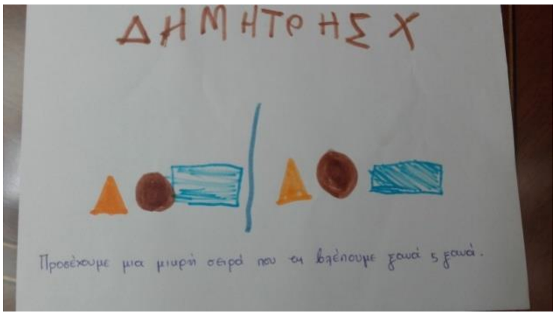
The use of a variety of manipulatives which supported different pattern structures, appeared to assist many children. In this way they elaborated a variety of different structures, since they were asked every time to recognize and formulate the small unit that was repeated, which element is first, which is last and the relationships among the elements. For example, when children had found the error of a given repeating pattern with ABBCD structure, they had to find firstly the structure focusing on the given material, and later to find the error. As for the growing patterns, children were asked to find and formulate the rule every time, which indicates the way that the pattern increases by a constant difference, as each row was one greater than the previous one. In a growing pattern, the children were given the first 3 rows and had to construct the 4$^{th}$ and 5$^{th}$ one. As the children were looking the pattern, some said: “It looks like a ladder” and a child pointed out: “It's like the abacus we have, in every row one marble more”. In this case, the children were helped conceive the rule of the pattern, as the manipulatives, reminded them other presentations with similar forms (ladder, abacus) (Figure 1).

**FIGURE 1**

*The growing pattern that the children say that looks like a ladder or an abacus*

The manipulatives also supported the teaching methodology, as the children focused on them and discussed with each other pointing at them. In many cases children who had managed to isolate the rule, showed it to other children by placing their palms vertically at the end of the rule and isolating it from the rest of the drawing, helping other children to understand the rule. For instance, a child said: “Now I understand how it's done! It is cut and continues again!”, which was also reflected in their conclusion, “cutting” with a vertical line at the point of the repetition of the rule (Figure 2).

**FIGURE 2**

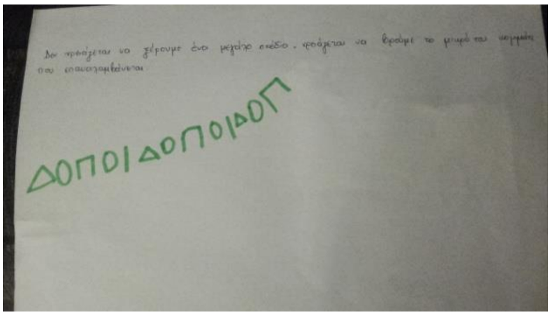



*“We watch out for a small order that we see over and over again”*

Yet the manipulatives underpinned the analysis of the generalization and the generalization questions. In each task the children had to identify the rule, analyze its components, and identify the relationships between them, focusing each time on the manipulatives. Then, they documented their action by showing on the manipulatives that they had in front of them.

Manipulatives also supported the generalization questions at local level and general level. In each task the children firstly recognized and formulated the rule of the pattern that they had in front of them. For example, when they were asked what they can do in order to remember the design that they had in front of them, they said “Look! We have to remember this small order that we see here and here and here”, pointing on the material and referring to the rule of that particular pattern. Afterward, without having manipulatives in front of them, the children ended up to more general conclusions about finding the rule. Examples referring to repeating patterns: “We watch out for a small order that we see over and over again” (Figure 2), “There is no need to know a big design, we only need to find its small piece that is repeated” (Figure 3). This conclusion was used as a strategy by the children in order to find a way to recall a design that was demonstrated to them and they had to reproduce it on their sheet of paper.

**FIGURE 3**



*"There is no need to know a big design, we only need to find its small piece that is repeated". This is the conclusion of the 3rd teaching that the children used in the 4th teaching*

Such an example concerns the growing patterns which is demonstrated in Figure 4 ("In each row we always put one more").

**FIGURE 4**


*"In each row we always put one more"*

The children were systematically recorded their conclusions by drawing which were posted in their classroom. In this way, they could use their drawings in order to cope with new pattern tasks.

## DISCUSSION

The results show that in the context of patterns the generalization ability can be developed from the preschool age (Papadopoulou, 2012), as children are able to conceive the rule of the pattern. As it can be inferred from the seven-stage mode, this conception depends on the recognition of the relationships of the rule which allows children to pass from a holistic and procedural understanding to a conceptual one. The recognition of the relationships is what helps children be aware of the structure of the pattern, have a deeper understanding of it. This finding is in the same line with previous findings which underline the significance of the development of relationships for the conceptual understanding (Hiebert & Lefevre, 1986).

The seven- stage model, representing the development of the rule, seem to have similarities with the four-stage model depicting the *development of a pattern structure* (Mulligan et al., 2005). We could say that the *pre-structural stage* of the four-stage model corresponds to stage 1 our model. The *emergent stage* corresponds to and is extended at stages 2a and 2b. The *partial structure stage* corresponds to stage 4a and the *structural stage* to the stage 5, while the stages 3 and 4b of our model do not correspond to any stage of the four-stage model.

It also seems that the procedural understanding (Gray & Tall, 1994; Haapasalo & Kadijevich, 2000) and the conceptual knowledge (Gray & Tall, 1994; Haapasalo & Kadijevich, 2000) appear already from the preschool age. In procedural understanding children correspond the elements of the repetitions one to one without being aware of the pattern unit that is repeated. Unlike the procedural understanding, in the conceptual one the children perceive the unit that is repeated and recognize its elements as well as the relationships among them.

With regard to the role of manipulatives to the rule generalization, the variety of the manipulatives helped children to engage in different pattern rules and to gain a deeper understanding of the concept, which is consistent with other findings (Jacobs & Kusiak, 2006).

The manipulatives also supported the generalization at local level, as the children were ending up to local conclusions about the rule focusing on the pattern that they had in front of them. Yet, they helped the children to develop strong representations and to arrive at a more general conclusion the pattern rule (Figures 2, 4) and develop a strategy in order to find it (Figure 3). This finding is also in line with other researchers who point out that manipulatives in patterns help to the generalization development (Blanton & Kaput, 2004).

In addition, the manipulatives supported the teaching methodology, as they enabled children to argue, document and justify their actions through reflection, allowing them to construct and maintain the new knowledge (Clements & McMillen, 1996).

Moreover, the use of the manipulatives helped children create their own mediating tools for new mathematical learning. As the children recorded and posted the general conclusion in which they were ended up each time, gradually a space with their conclusions (i.e. a poster) was created in the classroom, which the children could consult whenever they needed.

In this paper we have provided some evidence to the development of the pattern rule in early childhood. However, the research data derived from a class of 23 preschoolers, so the present findings cannot be generalized to other learning settings. Further investigation regarding both the evaluation of the seven-stage model of generalization in patterns and the role of mediating tools children produce through generalizing in their future performance would be useful.

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