The Role of Mental Argumentation in Mathematics vis-à-vis Property Perception and the Operational Mode

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ABSTRACT

In this article, mental argumentation concerning the doing of mathematics is characterized by that part of reasoning dealing with the perception of properties, in contrast to that involving an operational mode of thinking. In particular, the role of mental argumentation is argued in terms of semantics, holism, the usage of symbolism and what reasoning can be achieved without pen and paper. The discourse is based on a succession of mathematical tasks, stated in the text together with solutions and comments on how the solutions could be reached. What transpires is that there is a mutual dependence between mental argumentation and operational working, even though at first the two might seem to represent opposites. A partial account of the nature of this interaction is given.

KEY WORDS

Mental argumentation, property based thinking, meaning in mathematics, thinking globally, the role of symbols

RÉSUMÉ

Dans cet article, l'argumentation mentale au sujet de faire des mathématiques est caractérisée par cette partie de raisonnement qui traite la perception des propriétés, contrairement à celle qui implique un mode opérationnel de la pensée. En particulier, le rôle de l'argumentation mentale est discuté en termes de sémantique, holisme, l'utilisation du symbolisme et quel raisonnement peut être *réalisé sans stylo et papier. Le discours est basé sur une succession des tâches mathématiques, indiquée dans le texte avec des solutions et des commentaires sur la façon dont les solutions pourraient être atteintes. Ce qui transpire est qu'il y a une dépendance mutuelle entre l'argumentation mentale et le travail opérationnel, quoiqu'au début les deux modes de raisonnement pourraient sembler opposants. Un exposé partiel de la nature de cette interaction est présenté ici.*

Mots - clés

Argumentation mentale, pensée basée en propriété, sens en mathématiques, penser universellement, le rôle des symboles

Introduction

In this paper we consider what kinds of mathematical argumentation may be thought through 'in the mind'. The tightest criterion is to obtain an answer to a given mathematical task without recourse to pen and paper, or to any other physical medium of writing. However this criterion will sometimes be relaxed; in some cases we shall allow the drawing of diagrams and personal markings. These are taken nothing more than tools that facilitate the thinking done in the mind. In particular, they only have a referential role.

Why should we be interested in restricting the tools that we have at hand in this way? One compelling answer is that argumentation held in the mind is contrary to the common image of mathematics as being the manipulation of symbolism. Associated to this belief is that mathematics develops in an automatic mode devoid of meaning. However, if one is figuring something out 'in the head' things indeed are making sense in one-way or another.

If we take mental processing of mathematical tasks as a signal of sense making, it is important to investigate its nature, its limitations and what causes these limitations. One way to think about mathematics is that it provides channels of non-natural expression required to extend the limits what the mind could cope with before. What is produced by these channels may be transparent and can be incorporated into the existing naturalistic paths of thinking, or can be opaque in that some operational actions are utilized that are difficult to interiorize. This leads us to make the following definition. For mathematics, mental argumentation is thinking that is dominated through perceived properties rather than through a mode of applying accessible operations.

The distinction made in the definition above suggests a duality between meaning and functioning. Also when thinking about properties one would expect a global appreciation of the mathematical system that is presently at hand. Hence we touch on another duality, between holism and reductionism. A third dimension occurs when we think about the role of mathematical symbols, other types of mathematical constructs and strict logic. Does the usage of these give an indication that we are leaving the realm of mental argumentation?

In this paper I shall discuss the two dualities and the question brought up above vis-à-vis mental argumentation. In the next section I will review some related topics as they are treated in the mathematical educational literature. After this, the main body follows, which considers a series of tasks designed to illustrate several associated issues. The discourse is split into three sections, the first of which describes the form of the study as well as its rationale. For the second, the tasks are stated and analyzed separately; the forthcoming results are re-processed, assimilated and refined in the third such that the issues mentioned above are attended to in a coherent way. The description of the tasks are not based on fieldwork, but the author believes that they are representative of various kinds of situations that are open to argumentation held in the mind, or, to the obverse, present situations where it is obstructed. Hence this paper has a theoretical character rather than empirical one. Notice that we have retained here the term 'argumentation held in the mind' as well as 'mental argumentation'. The prior is described in physical terms; it stipulates restrictions on what is allowed to be written down. There is a tacit assumption that this is in accordance to the definition given for mental argumentation; within the concluding section I shall comment whether this assumption is completely grounded or not.

Background Material

In the following section we will discuss some of the topics mentioned in the introduction in the light of the existing educational literature.

The role of objects in mathematical thinking

The word 'mental' generally is used when a researcher wants to stress that the mathematics done is a result of the thought of the individual. This might seem superfluous in that all mathematics certainly is the product of the mind. Psychically, though, things are not so simple, evident from the Platonic viewpoint found in the philosophy of mathematics. The issue of Platonism in itself is perhaps not so crucial to the expert; it is useful to make an artificial detachment by regarding mathematical constructs somehow lying outside the psyche in order to think how further one might act on them (e.g., Brown, 1999; Burton, 1999). This theme in fact is much elaborated in the literature, in particular how the status of a mathematical 'object' is developed cognitively. This has a long history, from the work of Beth & Piaget on "reflective abstraction" in 1965, and of Greeno on 'conceptual entities' in 1983; in the 1990's several frameworks based on the idea of 'objectification' became prominent, from which probably Dubinsky's APOS theory (Asiala et al., 1996) and Sfard's reification (e.g., Sfard, 1991) have endured the test of time the better. Objectification has simultaneously both an 'interior' and 'exterior' nature; the mind has to recognize the status of a mathematical object, yet the object is conceived to assist imagining actions that could be profitably applied to it. If the actions made are not taken in a meaningful or natural way, this is an indication that the inner comprehension is diminished. There are reasons for the presentation of mathematics, i.e. the final version of one's work, to stress the exterior aspect. However, this has an unfortunate consequence in that the standard presentation given by teachers and textbooks promotes an image of mathematics as being self-generating. As a result, students output tends to reflect re-production rather than re-discovery or re-inventing. The whole hub of the movement of constructivism is centered on remediating this situation. In a rather controversial book by Lakoff and Nunez (2000), an attempt was made to reconcile the 'embodied mind' with theoretical mathematical developments through the idea of 'conceptual metaphor'. For other researchers (e.g., Tall, 2007) the term 'embodied' refers only to mathematical thought that can be made consonant to naturalistic understandings.

In the context of mental argumentation, lines of reasoning might have 'parts' that evolve through the management of applicable actions as well as 'embodied' aspects. For the prior, though, there is a restricted capacity for what can be held in the mind.

Property based thinking and the operational mode

The definition that we have made for mental argumentation supposes a distinction between property and operation. Note that this distinction is not made in absolute terms. This is partly because of the limited capacity to direct a string of operations in the mind.

On one level, a property is a logical consequence of a mathematically defined object (or system) that seems consonant and quite natural to how the object is perceived. In fact, the property itself can contribute to the perception, and the attainment of the status of 'property' is not necessarily instant. On another level, properties can be taken abstractly where the objects referred to do not retain a definite identity (Resnick, 1981; Rickart, 1996). So for us mental argumentation does not imply any differentiation between the abstract and the non-abstract.

In the operation mode, the passing from one state to another in the argument is effected automatically and so has no cognitive weight. (However, this might not be true for the synthesis implicit in the whole management of the argument). The operational stresses the obtaining of a result without necessarily conveying why the argument works (Skemp 1979; Jones & Bush 1996).

Meaning in mathematics, thinking globally and the role of symbols

Meaning in mathematics often is associated with conceptualization. Conceptualization is usually regarded to have two levels; first, how a concept is conceived initially and second, how this is represented as a mathematical definition, with the attending (natural) consequences and associations with other concepts. The latter level is commonly considered as a schema (as in APOS theory described in Asiala et al., 1996). A schema can be thought attached to a particular concept, or regarded rather as a knowledge base in problem solving generally (Marshall, 1995). Similarly, the notion of conceptual knowledge is characterized in Hiebert and Lefevre (1986) by relationships between mathematical objects. The significance of conceptual knowledge is that it promotes knowledge connectedness (e.g. Chinnappan & Lawson, 1996), so the memory is not over-loaded in making argumentation. A similar educational construct is the notion of 'compressed units' as explained by Barnard (1998).

When we talk about mental argumentation, we are primarily concerned with a process of obtaining a solution rather than conceptualization. How meaning can be accommodated within lines of reasoning? In this respect, Weber and Alcock (2004) compares semantic and syntactic proof productions, the former based on informal thinking and personal interpretations of the knowledge base, the latter on how to synthesize formal elements of the given system or situation. A similar duality is found in William and Black (1996), where meaning is associated with formative thinking in contrast to consequences that have only summative relevance. Weber and Alcock suggests that certain students have a penchant towards a certain type of mathematical reasoning. This is backed up in a paper by Duffin and Simpson (2005) that identifies four categories of 'cognitive styles'; in particular, the natural cognitive style links 'new mathematics to existing mental structures' and the alien cognitive style accepts 'new mathematical ideas as separate mental structures'. Another cognitive style, called 'coherence', displays a combination of the other two, suggesting that meaning can be extracted even from the most formal mathematics. This message is also found in Dubinsky (2000), and in Pinto and Tall (2002). Property appreciation does not only exhibit itself in intuitive thinking, as a property can be identified in the context of the examination of an abstract definition, its consequences and, most important, the apprehension of the significance of the consequences.

A parallel situation occurs when we think of mental argumentation vis-à-vis the contrast between thinking globally and analytically. The first impression might be that mental argumentation is only relevant to an integrated awareness of the system. However, in practice it is misleading to consider thinking on the global level as being separated cleanly from analytic examination. In Mamona-Downs and Downs (2008) it is claimed that a sense of structure acts as an intermediate role, and in this context analytic thought is endowed with a measure of meaning. In the same sort of spirit, Simpson and Stehliková (2004) talk of students' ability to appreciate structure through 'representational re-description'. In fact the term 'structure' might well be the most suitable medium to think about the cognitive difference between a property and a relation. In our view, mental argumentation straddles global and local thinking, whilst acknowledging that the degree of memory required might be greater for the prior.

A common term that occurs quite often in the mathematics educational literature is mathematization. This concerns constructing mathematical frameworks that helps the practitioner to extend what could be done mentally. But there is scope for mental argumentation to incorporate elements of these frameworks after they are extant. Perhaps the most ostentatious manifestation of mathematization is the usage of symbolism. The effective management of symbolism often needs a reference to what is being symbolized when the operational mode is compromised by a need of strategy making. In Harel and Kaput (1991) it is claimed that much mathematical notation acts as substitutes for conceptual entities, which is empowering but has its dangers. Gray & Tall (1994) introduced the educational framework of procept, where a symbol acts as a 'pivot' between a concept and the processes that it affords. The peril in not linking a concept with its corresponding processes can bring about dysfunction in effective reasoning, as expressed by the term 'proceptual divide'.

The Setting of the Examples and its Rationale

The context in which the examples were presented

The examples described below were deliberately chosen to reflect issues of argumentation held in the mind and mental argumentation. The points that they illustrate are not meant to be exhaustive in any way. The topic area of the questions will be diverse, and will also vary significantly in depth. I have presented the questions two times in talks with a mixed audience of mathematicians, educators and mathematics teachers, with the aim to discuss openly the role of mental argumentation in mathematics. The questions were given to the participants one by one, either verbally or written on the board. The participants were asked to answer the questions under the conditions that they did not write down anything and did not communicate with their colleagues. For each question, a varying period of time was allocated for them to think about the question and to get a solution. After the period ended, the participants were invited to express their solution and/or to comment on the question vis-à-vis mental processing. For the purposes of ease of expression, they were allowed to come up to the board to draw diagrams and fix some notation. They were requested only to write things on the board that was true to their original thinking. Finally for each question, I stated what I consider is the significance of the question as a paradigm for educational or cognitive issues, based on a mixture of the solutions given by the participants and those pre-mediated by myself. (A source of my prepared solutions was my previous experience in teaching most of the questions in a course on Problem Solving for undergraduate mathematics students over three years.)

Rationale

First, I explain what exactly motivated the presentation of the talk (made on two occasions with slight alterations), and second, why I believe a description of it has significance for mathematics education research?

For the first part of this question, I contend that although most teachers readily accept that mental argumentation is of crucial importance, they can be unsure about its exact nature and when, where and how it can be usefully implemented. Conversely, in what circumstances should it be avoided? The issue revolves around what can be afforded by mental argumentation and what cannot without recourse to analytic thought. The distinction between making a strategy and presenting a final solution is also an important factor. The talk was designed to open the issue to the participants. The device of forbidding the use of pen and paper served a way for them to gauge for themselves the potential and shortcomings of mental argumentation whilst they attempted the questions on their own. My commentaries, and suggestions from their colleagues, gave the participants an opportunity to re-assess the way that they work whilst doing mathematics. This experience, it is hoped, will influence the participants to introduce changes in their future teaching practices concerning mental argumentation.

Why to transmit an account of the talks? On one level, any tract in mathematics education should aim to encompass a contingent of teachers in its readership. In reading this paper, I believe a teacher would profit as if he/she had been present in one of the talks, as long as the reader is prepared to maintain a critical attitude throughout whilst making his/her own attempts and whilst reading the author's comments. Other readers might be mostly concerned about what this paper offers from the perspective of mathematics education as a discipline per-se. The significance of the issue dealt with can hardly be denied; but perhaps something should be said concerning the basis of the argumentation made in this paper. The aim is not to obtain data for cognitive analysis, but to give paradigms that illustrate aspects of doing mathematics in the 'mind'. The tasks and the particular solutions given to the tasks are deliberately made to make a premeditated point (but in a way that the participants' working can be represented as well). From a structural perspective, one can predict problems that students might encounter, or what would enrich their apprehension, by analyzing the 'make-up' of a problem. For such an approach empirical support is not used or required, but care must be taken not to exceed what can be reasonably extracted cognitively from a structural analysis of the task. There is a limitation here, but from another perspective there is also an advantage; stating a given solution can be made as a bench-mark with which other solutions can be compared, not on the basis of which is the 'best', but on the different kinds of demands that each exacts and the kinds of impressions that each evoke. Indeed, for several examples the commentary given explicitly includes such comparisons, and takes into account audience responses. In the same spirit, interpretations of stated solutions were not taken as being absolute; in fact a tacit policy of the paper is to modify assertions based on the consideration of one example by presenting other examples for which the same assertion has to be qualified.

The Illustrating Examples

Question 1 (1&1/2 minute)
Perhaps the most acknowledged type of mental argumentation is mental arithmetic. The participants were given 90 seconds to calculate 67×84; the writing down of anything was forbidden.

Remarks

Mental processing of arithmetic operations has never been important, and with the advent of hand-held calculators this is even truer. Some rare individuals seem to have extra-ordinary powers in this skill, but this does not follow that they will be good mathematicians. Capability in mental arithmetic would seem largely to depend on the ability of an individual's short-term memory to store, organize and then operate as suitable. There are different ways a problem in arithmetic can be done, and thus obtaining a solution is not necessarily a procedural act, but the practitioner is simply rationalizing efficient ways of mental working with thoroughly understood operations. Hence mental arithmetic is not a particularly rich medium to study mental argumentation; in the forth-coming questions I want to illustrate how mental arguments play a more crucial role in forming a solution strategy.

However I note that the mental processing of this question is likely to be organized in a different way than the standard multiplication algorithm. The way that I would tackle it myself would be:

Operate: $60 \times 80 = 4800$
Operate: $60 \times 4 = 240$
Add and then forget the two numbers above: 5040
Operate: $7 \times 80 = 560$
Add and then forget the two numbers above: 5600
Operate: $7 \times 4 = 28$
Add and then forget the two numbers above: 5628

Here one only has to remember at most two numbers at any particular time. Were the standard algorithm used, one probably would have problems in retaining the data when it is needed. The practice of 'carrying over' digits is one that is particularly suited to pen and paper work, but it difficult to process when calculating in the 'mind'. The mental processing above respects a more integrated point of view of the question than the one the algorithm offers; a point that will recur in other tasks encountered later.

In the talks, two or three participants out of roughly thirty were able to carry out the multiplication in the time allowed; they all took the approach above, but not necessarily in the same order of performing the actions.

Question 2 (2 minutes)
128 teams take part in a knock-out competition. How many games are there in the competition?

Remarks

For the reader who is not familiar with the term 'knock-out competition', it refers to a particular organization for conducting some sports championships. Basically the competition is arranged into successive rounds in which each team play another and only the winner qualifies to participate in the next round.

People who have not met this question before usually organize their modeling of the problem in the most obvious way, i.e. according to a structural understanding of the meaning of a knock-out competition. Hence they calculate the number of games in each round and then add. The strategy is easily formulated mentally but the action of the summation may strain the mental arithmetic. However by registering the numbers of games occurring in each round in terms of powers of two, we can avoid that task. We recall the identity that holds for any natural number n:

$$2^n + 2^{n-1} + \ldots + 1 = 2^{n+1} - 1.$$

When fitted into the current situation, n stands for the number of rounds, and n=6 in our case. And from this we can deduce that the number of games is 127. There is a more direct way to see this result, though. Every game has a unique loser; all the teams lose exactly one game except one team (the champion) who has not lost any game. Hence there is a bijective correspondence between the games and the teams excepting the champion, implying there are 128 – 1 = 127 games. This approach illustrates a role of mental argumentation that is far more creative than in the first. Here we are probing the given structure to discover relations and new perspectives, rather than trying to model what is immediately evident. Such radical switches in focus are often realized completely through mental argumentation.

It should also be noted that the bijective approach is more direct, in the sense that the answer is entirely conducted in the language of the context of the task, whereas the other approach involved an operation in a completely algebraic setting. Thinking in context is an important aspect, but not a characterization, of mental argumentation.

All participants indicated that their thinking was based on rounds rather than forming the bijection except one person in the second talk. (Interestingly, this participant was a pre-primary schoolteacher. Making correspondences is an important part in early learning, as is evident from the work of Piaget.)

Question 3 (5 minutes)

The symbol ${}_r C_n$ denotes the number of ways that r things may be chosen from n. Try to think of a justification that can be expressed verbally of the identity

$${}_r C_n = {}_{r-1} C_{n-1} + {}_r C_{n-1}.$$

Remarks

This equation was familiar to most of the audience; some knew an algebraic expression for ${}_r C_n$ but to use this to demonstrate the identity without recourse to pen and paper would clearly over-strain the retention of the symbolic manipulation required. Mental argumentation is more geared to fundamental understandings rather than to the handling of complicated calculations. We have to abandon numerical considerations and re-track to what the symbols signify cognitively in order to discover a natural relationship that explains the identity. We note that the left-hand side involves an enumeration of choices out of n things, whereas on the right-hand side enumerations out of n-1. This suggests that we should pick out one of the n things; call it x, say. On the left hand side x is not distinguished, on the right hand side it is. The right-hand side then is obtained as follows. The set of choices of k things out of n including x corresponds bijectively to the set of choices of k-1 things out of n-1. Similarly the set of choices of k things out of n excluding x corresponds bijectively to the set of choices of k things out of n-1. Hence the result follows.

Inspired from this example, I wish to make the following points about mental argumentation:

1. The mental process of 'discover a natural relationship' is meta-mathematical, in the sense that you are no longer occupied with finding the solution itself but you are engaged with realizing ideas that will enable a strategy or argument to be made.
2. Some of participants had some problems in following the mental argument presented above. Mental argument works very much on putting interpretations and meanings on mathematical entities and systems. To a degree such imputing of meaning will be personal. Hence the communication

of a mental argument depends on how readily the receiver accepts the interpretations implied in the transmitter's exposition.

3. The processing of a particular mental argument has a cognitive impact that helps it to be retained in memory. The result without the processing is more likely to be recalled inaccurately. Thus, for this question, it can be argued that it is better not to remember the identity, but every time it is needed to reprocess the argument justifying it.

There are several ways of obtaining the result by imbedding it in other environments. For instance, the participants who did make headway argued through coefficients of the expansion of the expression $(1+x)^n$. However, whenever such an approach was used it was supported from the memory of a past encounter of the task.

Question 4 (5 minutes)

N and M are different positive integers both divisible by 2^r but not by 2^{r+1} (where r is a natural number). Is there necessarily an integer between N and M that is divisible by 2^{r+1}?

<u>Remarks</u>

This question is one that might well arise in the context of a more complicated task that could be dismissed by the words 'it is clear that'. This would seem to suggest that the result is regarded to be transparent with a little mental argumentation. For our case we can process N and M as different 'odd' multiples of 2^r. But between any two different odds there must be an even, and so there is an 'even' multiple of 2^r between N and M. This argument carefully coordinates the formal and the informal. In particular it steers the semantics to fit with a familiar pattern that immediately yields what we want. The solution can be expressed formally of course, and we might even state an explicit number that fits the conditions, e.g. $N + 2^r$ that was suggested by several participants, but the informal argumentation above is highly accessible to mental processing and would capture the essence of any approach taken. Mental argumentation can be effective even on quite abstract systems if appropriate interpretations and associations are made. However note the slight abuse of language when we use the term 'an odd multiple of 2^r'; logically speaking it is absurd, but in mental argumentation if one knows and controls the intended meaning, there is no harm in employing it.

Question 5 (5 - 7 minutes)

Let n be an even natural number, f be a bijection $\{1, . . . , n\} \rightarrow \{1, . . . , n\}$. State a necessary and sufficient condition on f such that

$$\sum_{i=1}^{n} \left| f(i) - i \right|$$

is maximum.

A Possible Strategy

The question may be interpreted as expressing the maximal total displacement possible for a permutation of n objects (equally spaced on a line segment). If we consider in isolation the 'lower half' of the numbers, i.e. the set {1,2,…,n/2}, it would seem likely that the maximum contribution in displacement would be caused by mapping these numbers onto the higher half of the numbers, i.e. the set {(n/2)+1,…,n}. Moreover it is immaterial how this mapping is made. This choice forces us to map the higher-half of the numbers onto the lower half of the numbers, that we similarly assume yields the maximal contribution in displacement that the higher half of numbers could provide. Putting together any pair of such mappings would seem to yield the overall maximum.

In functional language then we conjecture that the maximum is achieved if and only if

$$f\{1,2,\dots,n/2\} = \{(n/2)+1,\dots,n\}$$
$$f\{(n/2)+1,\dots,n\} = \{1,2,\dots,n/2\} .$$

We indicate a way to show that the conjecture holds.

The summation to be maximized involves 2n numbers; each of 1,…, n appears twice. Of these numbers, n are taken positively, n negatively. Hence the maximum value of the summation ≤ (the biggest n terms added – the smallest n terms added)

$$= 2.n + 2.(n-1) + \dots + 2.\ (n/2 + 1) - 2.n/2 - \dots - 2.1$$

However, the bound is achieved exactly when f interchanges the bottom half with the upper. There are $[(n/2)!]^2$ permutations satisfying these conditions and with some calculations it can be easily shown that the value of the maximum is $n^2/2$.

Remarks

For this problem it is easy to conjecture a solution if you happen to think about it in the 'right' way, perhaps not so otherwise. Usually when I propose this task before an audience, the participants that do make some headway tend to approach the problem by constructing a particular bijection that seems likely to give the maximum (e.g., f(i) = = n-i+1), and then they can get somewhere towards the general answer by considering why the particular example works. This kind of semi-experimental and investigative work in the form of case consideration is of common practice, though perhaps it is not as ubiquitous as some literature would have it. The strategy outlined above is rarely realized by members of the audience. Its formation was strongly influenced by creating an interpretation for the quantity involved in the question. The interpretation encourages the idea why it is advantageous to regard f to act on a subset of its domain, rather than acting at the point-wise level. This exemplifies how mental processing, in the role of interpreting formal expressions, sponsors new perspectives allowing strategies that otherwise could have been overseen.

Once the significance of the permutations interchanging the 'upper half numbers' with the 'lower half' is realized, one might be satisfied that the key to the answer has been obtained and one might stop there. In the first presentation of the talk in fact I did this, but one member of the audience (University professor) did not believe the result so evident that a proof could be dispensed with. In the second talk, I decided to comply; this provided me with an opportunity to show that mental argumentation can have stages in its evolution. In the present question, we first form a conjecture, and after we continue to reason why the conjecture holds, in such a way that the reasoning could be compiled in the mind. This example might not be quite typical; the work following the formation of a conjecture tends to be more analytic in character and it may well be beyond what could be thought of unaided by written symbolism. However, usually the rudiments behind the thinking leading to the conjecture are reflected in any argument made to ascertain whether it is true or not (See Downs & Mamona-Downs, 2005). In this way we find an <u>influence</u> of mental argumentation on rigorous proof productions.

An operational mode of thinking tends to constrict the interest to the task given. Mental argumentation, as an active faculty of the mind, has an inquisitive factor, so it provides a milieu where investigations outside the tight sphere of the task environment may be raised. For example, for our case, it would be natural to ask what would happen if n is taken as odd rather than even in this question. I will leave it to the reader to consider how the argumentations and results would be affected.

Question 6 (3 minutes)
A seller leaves it to his customer to choose whether to take the sale deduction first and the tax second, or vice-versa. Which choice should the customer make?

<u>Remarks</u>
This question illustrates how limited mental argumentation can be if it is not allied with some mathematical structure. I have given it numerous times to various different kinds of audiences, but always the vast majority tries to argue intuitively and obtain an incorrect conclusion. However if they processed this problem by modeling both the reduction and the tax as quotients to be applied in a composite way to the original price, it would be clear by the commutativity of multiplication that the order is immaterial to the customer.

Question 7 (4 minutes)

A 3 × 3 square grid is filled up with 9 numbers such that the sum of the numbers in each row, column and in each of the main diagonals is the same. In the diagram below, only a few numbers are shown; the others are hidden. Find the number x that occupies the bottom and left corner cell.

		3
x	4	5

A Solution

We can fill three more squares as below:

x – 2	x – 1	3
	6	
x	4	5

Comparing in the two tables top and bottom rows, we have $2x=x+9 \Rightarrow x=9$.

Remarks

This task is a simpler variant of a well-known problem usually given the name the 'magic square'.

The first step of the solution is to deduce the number occupying the central square. This step nicely illustrates a contrast between mental argumentation and algebraic working in terms of the way it is achieved. Algebraically, you would set a symbol, say y, for this value, and make the equation: $x+4+5=x+y+3 \Rightarrow y=6$. Proceeding more 'in the mind', one notes that x is both in the ascending diagonal and the lower row, so the other two entries of both must sum to the same, and from this you immediately obtain 6. Is there any real difference? Cognitively, there is; by using the symbol y, the answer is obtained by 'blind' manipulation of algebra, whereas for the second 'method' the use of algebra is done in a frame of negotiation within the original context.

To complete this task, it is difficult to get an answer through mental argumentation; likely you would be forced to write down a little algebra to find x, as illustrated above. However a strategy can be formed completely in the mind; we have found the value of the central square, so one has two known values for the 'decreasing' diagonal and the middle column, allowing the first two entries of the top row to be expressed in terms of x. The third entry in the top, right square was a given that has not been used as yet; thus there would be a fair expectation that by equating the sum of the entries in the first row with the third we shall obtain an equation in x from which we can find x. The thought that some of the information given by the task environment has not been used as yet, so it is likely to be employed subsequently, is an act of executive control (Schoenfeld, 1992). This act does not have the character of applying a property nor of operational usage, so doesn't seem to fit comfortably with the duality suggested by our definition of mental argumentation, even though it is well within the compass of what could be held in the mind.

About a third of the participants were able to correctly process this task without writing anything down.

Assimilation of the Illustrating Examples

The questions that we posed suggest that the duality between meaning and functionality can be relative in several ways. For example, in question 1 a method is indicated for the multiplication of two 2-digit natural integers that differs from the standard one usually taught. This method, we contend, is better for handling calculations in the mind, but can become as routine as the taught one. Procedures can be carried out with an understanding of the properties that support them or can be followed blinded; for the former there is potential for adaption or generalization. On the other hand, it must be noted that as long as a procedure is functional there is no need to deviate from it. We pass to another point. The task environment is influential in imputing meaning. Whenever properties assumed from the context are employed, the argument is accomplished mentally. Modeling the situation mathematically steers the thinking away from the original situation towards a more abstract setting, where functionality comes to the fore. In cognitive terms, though, other issues may play a role. For question 2, we have indicated two solutions. In the first, slight modeling is made to transform how many games occur in each round to a summation of numbers whose form was simplified algebraically. In the second, the result was gotten by arguing completely within the context. For this reason, we could consider the second solution to be the more direct, and conveys more meaning. However, there is another way to look at this issue; because the first solution is more faithful to the original perspective of the statement of the task, then there is a different ground to gauge directness and perhaps also meaning. Furthermore, when we pass from the context of the original environment to algebra in the first solution, at both levels the thinking can be considered to depend on property appreciation. The effect of modeling into mathematical systems vis-à-vis semantics and mental argumentation is more complicated as one might first expect, and cognitive studies on this issue should be made in the future.

The tasks described in the previous section also indicate that a duality between global and local thinking is difficult to maintain without some qualifications. One aspect, again, is the form of the task environment; the wording may encourage an integrated examination or one that is more analytic in character. For example, in question 3, a proof of a certain identity (concerning the number of choices there are in picking a certain number of things out of all that are available) is asked for. The identity to be proved only involves the number of elements of certain sets. However, we described a mental argumentation that distinguishes a particular (but not specified) element to explain the result. Hence, the equation, whilst having the appearance of stating something on the global level, has also a local aspect in 'seeing' a way that shows the equation holds naturally. In contrast, question 5 deals with permutations maximizing a certain quantity. This at first sight might encourage an element-wise approach, but the clearest way to tackle the problem is to identify two subsets of the domain that whenever 'interchanged' yields the desired behavior. In this case, then, a task environment that first appears to be of an analytic character is in the end resolved by identifying in the 'mind's eye' a type of global transformation. Task 4 is described as one that might rise in a more elaborate argument but its explicit validation is omitted when passing from one line to the next. The reader is expected to fill up mentally the gaps in order to justify the written implications. This suggests global or integrated understandings are needed to support local aspects of an analytic exposition. Overall, mental argumentation involves not only thinking in terms of a whole, but also the management of switches between global and local perspectives, which is the essence of structural appreciation. This is in accordance to the definition of mental argumentation; properties can be perceived in two ways, first as being 'inherent' to the object or system considered, second as a 'working' consequence.

Making recourse to symbolism is a sign that operational tools are required beyond those that can be processed within mental argumentation. Question 6 is a strikingly simple example; the majority of people when asked whether it is better to take the sale reduction before or after the tax answer 'before'. Even when the mathematization of the situation is explained, some people persist doggedly to believe that there should be a difference in result. Question 7 illustrates how often and how quickly we have to resort to an algebraic mode of thinking divorced from the original context. These examples indicate that mental argumentation has severe limitations. However, we have already seen in other of the questions that symbolism can carry a semantic weight as well as a syntactic one. In question 5, an algebraic entity is interpreted in terms of an embodied notion (of total displacement), and this act of interpretation led to how the problem was most easily resolved. In question 3, it was required to establish a certain equation. The equation set out was symbolic. But it could have various readings; an equation between numbers, an equality between numbers expressed in a certain algebraic format, an one-to-one correspondence between certain sets, or the numerical equivalence that ensues when one computes a number of things in two different ways. The latter reading was extracted, yielding a mental argument that depends much on assigning properties expressed in naturalistic language. Certainly, then, there are cases where the process of mathematization can be reversed.

Finally, we comment on our implicit assumption that 'argumentation held in the mind' is a practical counterpart of the notion of mental argumentation. The latter is more expansive than 'argumentation held in the mind' because it allows for documentation of successive stages in a solution that might overload the capacity of memory if writing down was barred. The paper, though, was taken in the spirit that the context of 'argumentation held in the mind' was a particularly suitable vehicle to study mental argumentation. More disconcerting is a note in the commentary of task 7 that suggests that there may be situations where aspects of 'argumentation held in the mind' fails to be reflected in mental argumentation. The particular case involved an expectancy that a certain equation would yield the sought solution because the equation makes use of information not employed before. This thought clearly is processed in the mind, but its character is neither property based nor operational; rather it lies in the field of metacognition.

Conclusions

This paper considered the notion of mental argumentation in the context of mathematics. This notion is defined in terms of thinking about properties, set against operational working. In the introduction, we introduced the following issues that we thought were at the heart of the notion; does it represent that portion of mathematical thought that distils a sense of meaning, does it necessarily involve an overall appreciation of the solution, does it accommodate working based on symbolism and if so to which degree? Also, we wanted to check how well the definition of mental argumentation fits in with a more practical perspective, that is what can be done without the recourse to written working. In the bulk of the paper, a number of examples were discussed; what did they contribute to address the issues above?

The examples indeed drew out a distinction between property-based thinking and the operational mode. As there was not an empirical ingredient to the study, we were limited in what we could say explicitly about how students' cognition would differ between the two. However, our approach allowed us to predict situations that could be problematic and why. One factor here is the relativity of what is internalized as a property, and another how operational work needs guidance beyond the mere procedural. Further, the way that the task is expressed has an influence. Overall, the expected concord of mental argumentation with meaningful thinking holds only partially; for an argument to respect only a global perspective without recourse to 'active' symbolism is perhaps the exception rather than the rule. There are aspects of operational work that trigger property identification and structural appreciation, so 'touch' the heart of 'embodied' mathematics. Mental argumentation cannot cope with complicated reasoning based on the manipulation of symbolism or on analytic thought. On the other hand, it can briefly enter that realm in order to effect new structural viewpoints. It is important to research these interactions further.

The device that was employed to examine the extent that an approach in solving a task is processed mentally was how far the approach could be developed without recourse to pen and paper. This seemed a reasonable measure, if perhaps an extreme one. However, we noted that 'argumentation held in the mind' can have metacognitive aspects that seem not to lie comfortable with the duality of property perception and operational work. Despite of this, we still contend that researching what students / mathematicians can produce without pen and paper is a potent research medium to gauge the extent of property based thinking, and in the large the examples displayed in this paper bear this out. Educators potentially can learn a lot from this approach. But at the same time, it is important to exact any qualifications it may have.

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