Teaching suggestions for the measurement of area in Elementary School. Measurement tools and measurement strategies

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ABSTRACT

The present study deals with teaching the concept and measurement of area. 106 subjects of the 6th grade of Greek Elementary School measured the area of different kinds of shapes. The subjects were divided into two groups, an experimental group and a control group. In the experimental group, area evaluation was taught in a way that highlighted the conceptual characteristics of area measurement. The teaching intervention and the use of different measurement tools led to different measurement strategies. Moreover, the experimental group used more successful strategies than the control group.

KEYWORDS

Area Measurement, teaching area measurement, measurement tools, Elementary School

RÉSUMÉ

La présente étude traite de l'enseignement de la notion et la mesure de la surface. 106 sujets de la 6ème année de l'école primaire grecque mesurent la surface de différentes sortes de formes. Les sujets ont été divisés en deux groupes, un groupe *expérimental et un groupe témoin. Dans le groupe expérimental, l'évaluation de la surface a été enseignée d'une manière qui fait ressortir les caractéristiques conceptuelles de mesure de surface. L'intervention didactique et l'utilisation d'outils de mesure différents conduit à des stratégies de mesure différentes. En outre, le groupe expérimental a utilisé des stratégies plus efficaces que le groupe témoin.*

Mots-Clés

Mesure de surface, enseignement de la mesure de surface, outils de la mesure, école élémentaire

Introduction

Improving the teaching of mathematics is one of the main problems that both mathematical education researchers and mathematics teachers are faced with. One of its basic components and also a basic field of research is the tracing of problems in the comprehension of mathematical concepts, of the incomplete and at times distorted view students often take of mathematical concepts and their references. It is often proved that problems arising in the understanding of mathematical concepts are due to the use of traditional teaching methods, which overstress the familiarization with algorithms and underestimate the comprehension of concepts. One such example in Greek Elementary School is the teaching approach to the concept of area, which is the object of our study.

In general, area measurement of geometrical shapes in Elementary School has been widely investigated in numerous studies carried out in the field of mathematical education (e.g. Battista, 1982; Nunes, Light & Mason, 1993; Simon, 1995; Nitabach & Lehrer, 1996; Outhred & Mitchelmore, 1996; Kidman & Cooper, 1997; Brown, 2001; Kamii & Kysh, 2006; Zacharos, 2006; Silverman & Thompson, 2008; Clements & Samara, 2009). These studies, based on different theoretical perspectives, attempt to investigate the problems arising during the procedure of area measurement and to identify the causes of the difficulties faced by children. The lack of understanding of the specific features defining the notion of area measurement, particularly by Elementary School students, is often mentioned. This lack of understanding is usually attributed to the teaching approaches used in this specific topic.

The present study focuses on detecting appropriate teaching interventions that could contribute to an operational understanding of the area measurement procedure and to a more effective confrontation of relevant problems.

Theoretical Considerations

Most teaching approaches and techniques used for teaching mathematics nowadays are the result of many years of experience, incorporate a long course through history and convey diverse cultural influences. Prior to its development as a theoretical construction, Geometry was a human tool that determined the relations of humans with their surrounding space and environment. Even though the 'arithmetization' of mathematics has penetrated all areas of mathematics today, so that, in Geometry for example, "lines become endowed with a 'length'; the 'area' of a rectangle is the product of the lengths of its base and height, and this basic definition is extended to the area of more and more two-dimensional figures…" (Fowler, 1987; p. 8-9), nevertheless, up until the 2nd century B.C., Euclidean Geometry seems to have been different and completely non-arithmetized (Fowler, 1987). It must be noted that, at an early stage of its development, Geometry was related to the comparison of quantities, such as length, area, capacity, etc. In all these cases, measurement procedures are based on the similar physical characteristics of the quantities being compared or measured. For instance, the measurement of length involves the use of lengths, area measurement uses surfaces, and volume measurement uses three-dimensional objects. This is also observed in the way Euclidean Geometry deals with the issue of area measurement. The way Euclid compares surfaces differs significantly from the modern way of dealing with the issue, which is dominated by computational methods and the use of formulas. In Euclidean Geometry, the surfaces compared are broken up into parts so as to render them comparable (Bunt, Jones & Bedient, 1976). These comparison strategies are often based on a general proof method found in Euclidean Geometry and, in particular, in triangle equality criteria, namely the principle of *overlapping* (in Greek, 'epithesis'). The principle of overlapping can also be used expansively in the case of comparing and measuring surfaces, particularly in the early school grades, when the concept of area measurement is formed and its techniques are introduced.

In the case of measuring geometrical quantities, a significant effort during introductory teaching ought to be addressed at providing meaning to the measurement procedures (Van de Walle & Lovin, 2006; Zacharos, Antonopoulos & Ravanis, 2011). In any case, the distinction between the attributes present in the objects used as teaching materials must be made understood, and the attribute to be measured should be determined. For instance, in the case of geometrical shapes, the attributes measured could be related to length, such as the sides of shapes or their perimeter, or the focus could be placed on the surface occupied by the shapes. In the previous cases, different shape properties are seen, which correspond to different quantities. The distinction between attributes requires capacities that have not yet been acquired by children attending Elementary School. Thus, from a pedagogical point of view, it is more important to first develop the capacities to conceptually distinguish between attributes and choose which attribute is to be measured each time (Outhred & Mitchelmore, 1996; Kidman & Cooper, 1997; Brown, 2001; Zacharos, 2006).

In addition to the considerations mentioned above, the social-cultural approach (e.g. Luria, 1976; Vygotsky, 1978) claims that all higher mental functions are considered as culturally mediated activities. The role of the "tools" used in this mediation is considered crucial. A *tool* helps humans act on the object of their activity and is a mediatory means through which human external activity proceeds to tackle and control situations with which it is confronted. It also contributes to shaping human thinking and affects human behavior (e.g. Resnick, Pontecorvo & Saljo, 1997; Chassapis, 1999; Bart, Yuzawa & Yuzawa, 2008). Therefore, the measurement procedure must lead to the development of the ability to construct suitable measurement tools. These measurement tools may resemble established measurement tools, but they may also be arbitrary devices and children's constructions that facilitate the successful completion of the measurement. Each measurement tool has the advantage that it can be applied to a variety of cases, while it also mediates in and supports the construction of new knowledge (Stephan et al., 2001). According to Nunes and Bryant (1996), the measurement tools provided each time play a structural role in children becoming familiar with the concepts of measurement. This is because "the structuring of the children's action was not independent of the tool they had at their disposal in the problem-solving situation" (p. 308). Moreover, it is noted that the measurement procedure can be more effective when the dimensions of the measurement units for attributes such as length, area, or capacity correspond to the dimensions being measured (Nunes, Light & Mason, 1993; Zacharos, 2006).

Regarding the measurement of area in particular, one must stress the effectiveness of approaches that use surfaces as units of measurement, as for example when dividing surfaces into squares and overlapping measured surfaces with square units. Similarly, in other studies (e.g. Battista, 1982; Nunes, Light & Mason, 1993; Nitabach & Lehrer, 1996; Outhred & Mitchelmore, 1996; Kidman & Cooper, 1997; Zacharos, 2006) we see the deviation from traditional teaching approaches that are based on the algorithm *area=length x width* (or *base x height*). The basic assumption in these studies is that the measurement procedure can be more effective when there is a correspondence between the dimension of the measurement tool and the dimension of the measured surface. In the aforementioned studies, particular emphasis is placed on the difference between the measuring procedures of length and area. It should be mentioned that, while length is measured directly, area is calculated indirectly with the use of linear attributes that are inserted into the formula of the area. Problems faced by students in understanding area measurement are attributed to this indirect way of defining the area. For this reason, it is deemed best to use two-dimensional units (such as the square surface) as measurement units for surfaces. Finally, it is stressed that the design of the study and the conditions of its implementation affect the strategies selected for dealing with problems (Nunes, Light & Mason, 1993).

The present research paper aims at tracing appropriate ways for teaching area measurement in the early school grades. More specifically, it investigates whether teaching interventions adopting qualitative and more conceptual approaches for area measurement, such as overlapping, along with measurement tools that maintain similar physical characteristics to those of the measured attribute, such as two-dimensional measurement units, can lead students to adopt different and more effective area measurement strategies.

Method of Inquiry

Design and sample

The collection and analysis of empirical data in this study are based on an experimental procedure which allowed us to compare the performances of two groups of pupils attending the 6th grade of Greek Elementary School. The experimental group (E.G.) consisted of three classes with 56 students in total, and the control group (C.G) consisted of three classes with 50 students in total. The sample was drawn from the 6th grade because it is at this grade that the teaching of area measurement of plane figures is concluded. In addition, students were taught for using formulas to calculate areas from basic geometrical shapes, such as rectangles, parallelograms, triangles and trapeziums, from the 4th and 5th grades. In all cases, students were taught to calculate the area of plane figures using the formulas and not on given emphasis on conceptual approaches. For instance, the process of area measurement for a rectangle, was focused on finding the formula, area = length $\times$ width. In this way, we were given the opportunity to assess the knowledge of students graduating from Elementary School and to investigate the strategies they followed for solving the tasks given to them.

The students in both investigated groups came from the same schools, where each grade is divided into classes based on alphabetical order. The experimental plan used is known as the "post test-only control group design" (Cohen, Manion & Morisson, 2004), and the difference between the two groups is what we call our independent variable, which is inserted into the experimental group in order to measure its effects. In our case, the independent variables are the teaching intervention and the measurement "tool" used when teaching the experimental group.

Teaching intervention

The subjects of the experimental group were submitted to a teaching intervention based on a lesson plan which was provided to the class teachers and which focused on the conceptual characteristics of area measurement (see Appendix A).

More specifically, the issues relevant to our current study that were brought forth during the course are the following: Firstly, *the Euclidian method for the comparison of surfaces.* It is noted that when Euclid wanted to show that two figures had equal areas, he proved that one of them could be divided into such parts that, if properly reconstructed, could create the other figure (e.g. tasks I in Appendix A). This procedure is called the "additivity axiom" (Wagman, 1975; Freudenthal, 1983). Secondly, *the principle of overlapping ('epithesis').* Euclidean geometry uses as a general proof method the principle of overlapping. According to this principle, the two attributes are compared by applying one onto the other. Here we use an expansive interpretation of the notion of overlapping that also includes the measurement/overlapping of surfaces (e.g. tasks II in Appendix A).

In all cases, students were taught by their regular teachers. The experimental group teachers participated in a training session and were informed in detail as to the purpose of the survey, the historical evolution of the concept of area, and the aforementioned procedure of overlapping in Euclidian Geometry. During classes, the teachers provided the students with elaborations as well as guidance in using the measurement tools; it was ascertained that the teaching of the experimental group proceeded in compliance with the purpose of the study. Meanwhile, the control group teachers taught area measurement according to the conventional way suggested by the Greek school curriculum, following to the schoolbook normally used. In this case, students became familiar with the use of formulas and not with approaches that maintain the conceptual characteristics of area measurement.

The interview and the research tasks

Two weeks following the teaching of area measurement, all subjects participated in a semi-structured individual interview, where they were asked to solve surface comparison and measurement tasks different to the ones that had been used during teaching (see Appendix B). In the present paper we present four tasks. In the first task, students are asked to calculate the area of a rectangle. In the second task, they are asked to calculate the area of an irregular geometric figure. The measurement tools used by the two research groups are different in the first two tasks. We have mentioned in our theoretical framework that the premature deployment of arithmetical methods in surface comparison and measurement is responsible for the conceptual problems causing measurement ineffectiveness. In teaching the experimental group, we promoted a different approach to area measurement. We tried to include this approach in the measurement tools given to the experimental group students. A square unit of 1cm side was drawn onto the worksheet (see

Appendix B) and used by the students of the experimental group. Where there was difficulty in 'transferring' the square unit onto the area measured, the students were given a small piece of cardboard measuring one square centimeter. In addition, students of the experimental group used an unmarked ruler (i.e., with no measuring ticks) as an aid in drawing lines, while control group students were given a regular marked ruler.

For the following two tasks, both groups are provided with the usual marked ruler. In the third task, we investigate whether the teaching imparted to the experimental group students helped them to better understand the conceptual content of area measurement. More precisely, we investigate whether the students understand that the arithmetical expression of area in cm^2 actually expresses the number of square centimeters that can "fit" into the measured surface. Finally, in the fourth task, two figures with non-equal areas (a 4cmx6cm rectangle and a parallelogram with sides of 4cm and 6cm) yield areas of equal absolute value due to the different measurement units used. Our aim is to investigate whether the students comprehend the significance of the measurement unit used in area measurement.

The interview also included a 'teaching' aspect. That is, the presentation of each new task followed the answer of the previous one, an answer given by the student either alone or with the assistance of the researcher. The tasks used were either drawn from relevant studies of student knowledge assessment or invented by the researchers. In all cases, the symbolic and verbal language used in schoolbooks was adopted. In order to ensure the constructional validity of the suggested tasks, we selected them after a series of empirical tests performed on students who were not included in the final study sample. Both student groups wrote in pen so that we could later retrieve all their notes.

The collection of empirical data was achieved through an individual interview with all students included in our sample. The interview took place in a private room provided to us by the school. Only the researcher and the student were present at the session. The students were requested to fill in a worksheet with the same tasks addressed to both research groups (Appendix B). If during the course of the interview some issue of special research interest arose, further clarifications were requested. The interviews were recorded.

Findings

Construction of a tool for measuring length

The difficulties faced by the experimental group in the measurement of the rectangle's area and the reproduction of the unit surface led students to the invention and construction of a measurement instrument for length.

We distinguish between the following two categories:

(a) *Use of a ruler*: A group of subjects (approximately 64% of the subjects of the experimental group) first place the unmarked ruler along one side of the square unit and then make a mark with their pencil on the ruler to indicate 1cm. This length is then reproduced along two consecutive sides of the figure measured. This procedure assumes a variety of forms. For example, one subject marks on the ruler the length of the rectangle side and then places the ruler along one side of the square unit, where the subject marks how many times this side fits into the length of the rectangle side. The same procedure is repeated for the other side of the rectangle. Then the subject squares or multiplies and gets the area through use of the formula. Other subjects construct a more complete tool for the measurement of length. With the assistance of the square unit, they make ticks on the ruler for 1, 2, etc., up to 6. Then they mark the two consecutive sides of the rectangle.

(b) *Use of a square piece of cardboard*: In this case, the square cardboard piece is used as the length unit for drawing the two consecutive sides (this is the case for approximately 26% of the subjects of the experimental group).

Strategies for the measurement of area

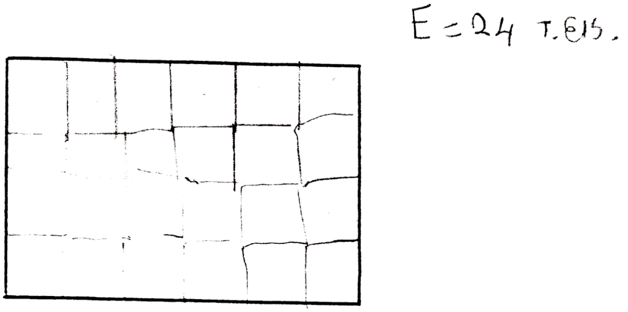
We assessed the effect of the measurement tools available for students to measure surfaces. We were interested in two aspects of this effect: the effectiveness of the measurement and the possible variations in the strategies used which were imposed by the use of these tools. We found that the difference in the teaching of the two research groups and the use of different measurement tools 'led' students to adopt different area measurement strategies. Indeed, students of the experimental group mainly resort to overlapping strategies or divide the measured surface into unit surfaces, while students of the control group mainly use the formula *area=base x height*. Furthermore, these students often seem to possess an incomplete understanding of the concept of area, which seems to be due to the traditional teaching method overstressing the use of formulas. The analysis of the findings that follows will illustrate these issues.

Task 1

In this task, students were asked to measure the area of a 4cmx6cm rectangle. They mainly used the following strategies:

(a) Division of the given area into unit squares and enumeration of the squares. This strategy is used mainly by students of the experimental group. Some of the subjects used the unmarked ruler with the 1cm tick in order to reproduce the unit square, while others reproduced the square cardboard piece onto the surface of the rectangle (e.g. Figure 1).

FIGURE 1



Overlapping with the use of the unit square (area=24cm^2)

This is usually a procedure of counting imaginary squares, as for example in the case of a subject who uses the following mental counting of squares: "Four and four more is eight. Eight and four more is twelve… and another four is twenty four". It is a procedure of unitizing, as analyzed in Wheatley and Reynolds (1996).

(b) U*sing the area formula*. This category includes subjects who use the square cardboard piece as a length unit in order to draw two consecutive sides and then implement the formula, or who use the unmarked ruler bearing the ticks made with the assistance of the cardboard piece, or – in the case of the students of the control group – those who use the regular marked ruler to draw two consecutive sides. For example, a subject counts six imaginary squares horizontally and four vertically. During multiplication, however, the imaginary square at the corner is measured only once, and therefore the result is: area=6x3=18cm^2. This is a mistake made even by prospective teachers and is mentioned by Simon (1995).

In this category we have also included some 'mixed' strategies. Although these strategies include the division of the surface into squares (i.e., division of the entire surface of the rectangle based on its two consecutive sides), the *area=length x width* formula is also implemented. For example, after having divided the surface into squares, a subject finds:

S (Subject): The area is one, two, three, four (for the list of squares) and one, two, three, four, five, six (for the line). Four times six equals twenty-four.

Regarding investigation of the second part of our hypothesis, according to which students of the E.G. mainly use division into squares and enumeration as a surface measurement strategy, while those of the C.G. use formulas, Table 2 describes the arithmetical data regarding the strategies followed. Table 1 presents the difference in measurement strategies selected by the two groups, which is statistically significant (x^2=32.680, p=0.000).

TABLE 1

Strategies for area measurement in Task 1

	Experimental Group		Control Group	
	N	F%	N	F%
Division into squares and enumeration	30	53.5	1	2
Use of formulas / 'mixed' strategies	23	41	47	94

Students' performance. The data in Table 2 show predominance of the E.G.'s performance with a statistically significant difference (x^2=9.169, p=0.002).

TABLE 2

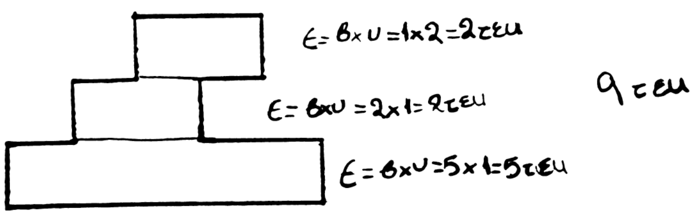
Students' Performance in Task 1

Experimental Group				Control Group			
Success		Failure		Success		Failure	
N	F%	N	F%	N	F%	N	F%
54	96.4	2	3.6	37	74	13	26

Task 2

Successful strategies: Students who succeed in the measurement of the surface shown in Figure 2 (Appendix B) follow mainly two strategies. In the first strategy, subjects divide the surface of the figure into squares and count the square centimeters formed (78.6% of the subjects of the experimental group and 4% of the subjects of the control

FIGURE 2



The strategy of division into rectangles group). In the second strategy, subjects divide the surface into rectangles and then add up the areas of the rectangles formed, which they calculate through use of the relevant formula (19.6% of the subjects of the experimental group and 54% of the control group). This is, for example, the case of Figure 2.

Unsuccessful strategies: Subjects who fail to measure the surface in Figure 2 can be placed in either of two groups. The first group comprises subjects who use the formula that yields the area of a rectangle. These subjects, affected by the formula *E=base x height*, try to use it even in cases where difficulties are clearly visible. This happens, for example, in the case of a subject who does divide the surface into squares using the marked ruler, but nevertheless still tries to use the formula *area= base x height*:

S: The base is five centimeters… Now what's the height?

When using these formulas, a large variety of choices is observed in the selection of "length", "width", and "height". Some subjects use the formula area=5x4 or area=5x5. The number 5 corresponds to the "base" of the figure, while 4 is the sum of the remaining squares apart from those of the base and 5 is the sum of the external unit sides (see Figure 3). For example, one subject confused the notion of area and its arithmetical expression. After having divided the surface into squares using the square cardboard piece, the subject uses the multiplication 5x4=20cm^2 (where 5 is the number of squares along the base and 4 the number of the remaining ones).

R (Researcher): So, what's its area?

S (Subject): Twenty-four.

R: How many squares (such as the square cardboard the subject is using) can fit into the surface of the figure?

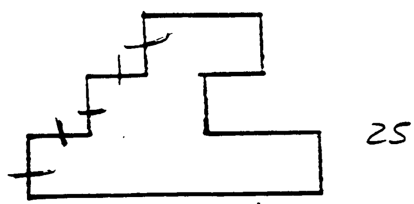
S: Four plus five, nine.

R: So… what's the area of the shape?

S: Oh! I've mixed it all up! Twenty… Nine… Nine squares can fit, but if we multiply them… then twenty can fit!

Another group of subjects uses the formula E=5x3, where 5 is the length of the "base" and 3 corresponds to the "height". The definition of "height" varies here. In some cases, it is what is formed when drawing a line perpendicular to the "base". In other cases, "height" is the sum of the three vertical units.

FIGURE 3



The area=width x height strategy (5cmx5cm=25cm^2)

The second group includes subjects who think that the figure has no area or that there is no way of calculating it. The main justification is based on how complicated the figure is. For example, a subject states that:

S: I don't think it has an area!

R: Why?

S: Because its shape is strange!

Another subject states that:

S: It doesn't have an area.

R: Why?

S: Because it has too many lines, too many sides!

Finally, there is a group of subjects who state that they are not aware of the procedure for calculating the area of the figure and they confuse the notions of area and perimeter.

Regarding the investigation of the second part of our hypothesis, according to which students of the E.G. are expected to use division into squares and enumeration as their main surface measurement strategy, while those of the C.G. are expected to use formulas (in all tasks presented here), the data in Table 3 shows that our hypothesis is confirmed. The variation in measurement strategies between the two groups is statistically significant (x^2=38.063, p=0.000).

TABLE 3

Strategies for the measurement of surface in Figure 2

	Experimental Group		Control Group	
	N	F%	N	F%
(I). Division into squares and enumeration	44	78.6	2	4
(II). Use of formulas	11	19.6	27	54

Task 3

Through this task we aim at investigating whether the students comprehend to which physical quantity the arithmetical expressions of area derived from the formulas correspond, and whether there is any variation in the degree of comprehension between the two research groups. In this task, students are asked to calculate the area of Figure 3 (Appendix B) using the relevant formula. In this case, students of both the experimental and the control group use a regular marked ruler. In cases of failure, students are reminded of the relevant formula (*area=base x height*) and all teaching interventions necessary are performed in order for the interview to continue smoothly. Students are then asked how many times the square unit (which has been drawn onto the worksheet) can fit into the surface of the rectangle. The students' answers were grouped as follows:

(a) A large umber of students (60.7% of the experimental group students and 30% of the control group students, see Table 4) provide the correct answer directly, with no need for resorting to the figure.

(b) A second group of students have difficulties in providing a direct answer and need to resort to the figure. These students are divided into subgroups:

In one subgroup, students first count three vertical and five horizontal square units and then resort to the formula 3x5=15, as does, for instance, the following student:

S: The little square fits five times into the base and three times into the vertical side, so three times five is fifteen.

Another subgroup of students counts imaginary square units, either successfully or unsuccessfully. For example, one student produces the correct arithmetical result using the formula, but nevertheless states that the square unit fits twelve times

R: Why? How did you come up with that?

S: I (mentally) divided it into little squares and found 4 here (indicates horizontal direction) and 3 here (indicates vertical direction)!

Another subgroup comprises students who divide the area of the rectangle by the area of the square unit, as in the case of the following two subjects:

- "Fifteen times divided by one… (equals) fifteen times!"
- "One time one equals one square centimeter (the area of the square unit). Now we divide fifteen by one, fifteen divided by one equals fifteen. It fits fifteen times!"

Another subject uses a variation of the previous method:

R: How many times does this little square fit into the rectangle?

S: …Seven and a half.

R: Why seven and a half?

S: I figure one time one equals two (the area of the square unit). Fifteen divided by two equals seven and a half times!

Another subgroup includes the students who used a variation of the previous case: in this case, the division used is 15:4, where 4 represents the perimeter of the square unit. Below are some indicative responses:

R: Why seven?

S: I add the sides (of the square unit), one and one and one and one equal four. It's fifteen divided by four (claims this division yields 7)!

Another subject suggests:

S: We must find the perimeter of this (square unit) and divide it (means to say 15:4).

Finally, another subgroup of students confuse area with perimeter. In this case, students either count the square units found along the four sides of the perimeter

(5+3+5+3) and claim the square unit fits 16 times, or they count the square units found along two consecutive sides of the perimeter and claim the square unit fits 8 times.

(c) This is the case of students who provide idiosyncratic answers with numbers that are neither related to the task's data nor justified. This group also includes students who state they do not know. One such is the student who answers:

S: I don't know. We should calculate it!

TABLE 4

Strategies for the measurement in Task 3

How many times does the 1cm^2 fit into the 3cmx5cm rectangle?	Experimental Group N	Experimental Group F%	Control Group N	Control Group F%
I) 15 (without resorting to the figure)	34	60.7	18	36
II) After resorting to the figure				
IIa 3x5=15	2	3.6	5	10
IIb Mental enumeration	12	21.4	8	16
IIc 15:1=15	0	0	3	6
IId 15:4	0	0	1	2
IIe 16 or 8 times	3	5.4	7	14
III) Do not know/other answers	5	8.9	8	16

The students' answers to the previous task show that the majority of the control group students do not comprehend the physical quantity denoted by the arithmetical expression of the rectangular surface's area. Furthermore, the variation in measurement strategies between the two groups is statistically significant (x^2=5.5049, p=0.018).

Task 4

This task, as did the third one, intends to investigate whether the students in this study understand the physical quantity denoted by the arithmetical expression of area. More precisely, our intention here is to investigate whether the students are aware of the fact that the number derived from the area measurement of a figure is not an absolute quantity, but rather that it relates to the unit of measurement used each time. For example, the areas of Figures 4A and 4B in Appendix B are given by the same absolute value (24 units), but are in fact two different quantities, since the measurement units are different. In the first case, the measurement unit is the square centimeter, while in the second one the unit is a rhombus whose sides are 1cm and whose angles are equal to those of rectangle 4B. After the students are asked to calculate the area of the two figures with the respective units (and in case of difficulty, also with the intervention of the researcher), they are asked whether either of the two figures has a larger area or not.

The students' answers (see Table 5) were grouped into the following categories:

(a) One category comprises the students who gave the correct answer, i.e. that area(A)>area(B). These students depend either on calculations using the *area=base x height* formula or on sensory intuition ("because that's how it looks").

(b) Another category includes students who claim that area(A)=area(B). This claim is based largely on the equality of the arithmetical expressions of area derived using the two different measurement units. For instance, one subject claims that surfaces A and B are equal because their area is expressed numerically by the same number.

R: So this little square (the square centimeter) has the same area as this rhombus (unit rhombus b)?

S: Yes! But it's a bit skewed, so it looks different. But it's the same.

Some students in this category seem affected by the general "topological" (Piaget & Inhelder, 1956) features of the figures (e.g. "They have equal areas because if I straighten this (B) it will be the same shape as A") and/or the equality of their perimeter (e.g. "they have the same length and width").

(c) Students of another category claim that area(A)<area(B). These students base their claim on sensory perception data ("because that's how it looks").

(d) Finally, some other students hesitate to answer because they have doubts as to their judgement, or say they do not know and give up, or, finally, give idiosyncratic answers with no justification.

The data in Table 5 show that the answers given by students of the experimental group are more successful and the difference in successful answers is statistically significant (x^2=5.019, p=0.025).

TABLE 5

Comparison of areas for figures A and B

Comparison of areas A and B	Experimental Group N	Experimental Group F%	Control Group N	Control Group F%
area(A)>area(B)	12	21.5	3	6
area(A)=area(B)	37	66	45	90
area(A)<area(B)	1	1.8	0	0
Do not know / other answers	6	10.7	2	4

The case of the fourth task highlights the students' confusion regarding perimeter and area, as well as their incomplete understanding of the significance of the measurement unit in area measurement. The following conversation excerpts stress the previous points.

In the beginning, the subject confuses the area of the figures with their perimeter, which is 20cm for both figures:

R: How many times does this little square (the square unit) fit into figure A?
S: Twenty times.
R: And how many times does this rhombus (the unit rhombus) fit into figure B?
S: Twenty times.
R: If we wanted to calculate the area of figure A, what would it be?
S: Four times six (equals) twenty-four square centimeters.
R: So how many times does this little square fit in here figure A)?
S: Twenty times… Twenty-four times, because this is the area.
R: Now let's look at figure B. How many times does this rhombus (the unit one) fit into figure B?
S: Twenty-four times.
R: Now if you take a close look at figures A and B, does any of them seem to you to have, to occupy, a larger surface, or not?
S: No, are equal.
R: Now let's try to use the ruler and the formula to figure out the area of the second figure.

The subject uses the formula for the trapezium in order to calculate the area of figure B (Area=$(6+6) \cdot 3/2=18$ cm^2).
R: So does either of the two figures have a larger area or not?
S: They have the same area.
R: And what is it?
S: Eighteen!

Discussion

In the present study we tried to assess the contribution of teaching when adopting an approach towards the concept of area and the measurement of surfaces that emphasizes their conceptual aspects. Our basic principle regarding measurement was that the tool mediating in the measurement must have the same dimensions as the measured quantity, which is to say that the tool used for the measurement must maintain the physical characteristics of the measured quantity. We tried to apply this rationale to the measurement tools made available to, and used by, the experimental group students of our study.

It was indeed found that the tools suggested to the students of the experimental group for measuring areas, as well as the teaching regarding the use of these tools, "suggested" to these students the particular strategies they later adopted and also affected how they organized their activities (first and second task). In the case of the experimental group, the positive contribution of teaching included teaching of the "overlapping" procedure for surface measurement, which refers to practices of quantity comparison in the context of Euclidean Geometry. These practices show the historic development of these notions, and their integration into the teaching process proves particularly fruitful.

At another level, the present research sought to trace aspects of the students' incomplete understanding of area measurement. As shown by the findings, many students, particularly from the control group, fail to comprehend the physical quantity denoted by the arithmetical expression of area. It would seem that, in the procedure of area calculation through the *area=length x height* formula, the algebraic characteristics of the procedure take on the leading part, which results in the students not understanding that the arithmetical expression of area has a physical content related to the measurement unit used each time. Moreover, the role of the measurement unit in area measurement is not adequately understood by the majority of the students (task four, Table 5). Students have difficulties in understanding that equality of the arithmetical expressions of the two figures' areas does not necessarily imply equivalence of the two surfaces, and that the measurement units used should also be taken into account. In the case of the fourth task, it seems that the two figures' "topological" morphological features, along with the equality of their perimeters, affect the students' judgment as to the areas of figures A and B, as well as their judgment as to the respective measurement units.

Furthermore, it was found that students who failed in the tasks of surface measurement were persistent in the use of strategies whose generalization leads to failure. These strategies bear the characteristics of a teaching obstacle and are affected by certain basic school practices. This observation, along with the study of the strategies that led to failure, may help us draw some useful conclusions as to the pedagogic strategies used by students in measurement (Brousseau, Davis & Werner, 1986; Borasi, 1994; Ryan & Williams, 2007). For example, generalization of the use of the *area=base x height* formula (e.g. Figure 3) shows the problems attributed to premature arithmetization in area measurement (Zacharos, 2006).

Another error observed in the students' answers, though not as frequently as the "*area=base x height*" error, is the "*area=base+height*" error. For instance, in the first task, some subjects added together the lengths of two consecutive sides of the rectangle and derived area=6+4=10cm^2, while in the third task they derived area=5+3cm^2. According to certain interpretational approaches (e.g. Anderson & Cuneo, 1978; Leon, 1982; Lautrey, Mullet & Paques, 1989), two different psychological learning mechanisms operate in the "base+height" strategy. First the student answers based on the entire shape of the figure, taking into account the dimensions of the "base" and "height". A second mechanism is related to the child's total focus on the perimeter. Here the child "scans" the perimeter of the figure. Of course, according to the aforementioned researchers, these two mechanisms cannot be easily distinguished, since, for example, the perimeter is two times the height+width. The "base+height" rule may be considered as an initial perceptual procedure of area calculation that requires a higher cognitive ability than does focusing on only one dimension. However, the mechanism of understanding the notion of area is cognitively more complex and is based on the method of analogy. In the case of area, the figure is (imaginatively or actually) covered with squares, and the number of squares provides an approximation of the area. As a psychological analogy, we could consider that the figure is "scanned" by an imaginary square unit and the total number of imaginary squares provides an approximation of the area (Anderson & Cuneo, 1978; Leon, 1982; Lautrey, Mullet & Paques, 1989). As has been mentioned above, this kind of error shows the need to apply more qualitative approaches in the teaching of surface measurement.

Although the procedure of overlapping the measured surface with the measurement unit supports the conceptual understanding of area measurement by Elementary School students (a fact demonstrated in our study by the answers of the experimental group students), on the other hand it seems difficult for students of this schooling level to understand how the counting of square units or rows of square units can lead to an arithmetically equivalent algebraic expression of area, such as the *area=base x height* formula. The psychological background behind students' difficulties in understanding the transition from square units to the use of the *area=base x height* formula is mentioned in the work of Piaget, Inhelder and Szeminska (1960), as well as in more recent studies, such as Kamii and Kysh (2006). Piaget, Inhelder and Szeminska (1960) point out that it is difficult for students to comprehend how the lengths involved in the area formula can create a surface. One way to understand that the product of two lengths can generate a surface is to consider a thin vertical strip, whose length is equal to that of one of the two perpendicular sides, "sliding" along the other side. In this case the surface is "scanned" and a rectangle is produced. The surface production model described here is found in mathematics in the first approaches of the concept of the integral. In fact, the historical roots of the integral, before its rigorous definition by Cauchy and other mathematicians following the development of calculus in the 19th century, is based rather on an intuitive perception which relies on geometry and which was successfully developed quite early on (1350) by Nicole Oresme (Boyer, 1949). The difficulty encountered by young students lies in understanding the notion of an infinite number of infinitesimally thin strips scanning a surface along the length of one of its sides (Kamii & Kysh, 2006; Piaget, Inhelder & Szeminska, 1960).

To conclude, the aim of the present study is to contribute to an on-going effort in mathematics education to render school mathematics more comprehensible to students, especially in regard to measuring areas. While the teaching approach towards area measurement, particularly in tasks where the surfaces measured are irregular geometrical shapes (see second task), provided positive results for the experimental group of this study, this finding alone does not allow us to generalize on the effectiveness of the specific strategy. What is more, it stresses the need for targeted teaching interventions, in order for the children's transition from overlapping practices to formula use in area measurement to be followed by the necessary comprehension of the physical quantity denoted by the concept of area.

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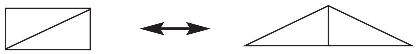
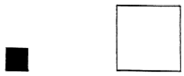
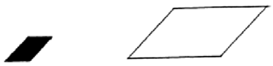
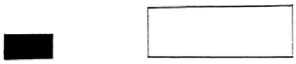
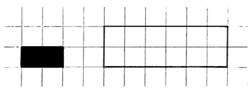
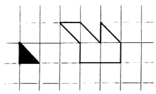
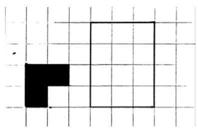
Appendix A

(Indicative activities related to comparison and evaluation of area, carried out during the teaching process)

I. *The Euclidean method of area comparison.*
Students were asked to compare the two areas in Figures Ia, Ib, and Ic.

Figure Ia	
Figure Ib	
Figure Ic	

II. *The Principle of 'Overlapping'*. Students were asked to measure the areas of figures IIa, IIb, IIc, and IId through the strategy of 'overlapping', using the given unit.

Figure IIa	
Figure IIb	
Figure IIc	
Figure IId	
Figure IIe	
Figure IIf	

APPENDIX B

(What is required in all tasks is the measurement of the area of the figures drawn)

Task 1: Students were asked to measure the area of a 4cmx6cm rectangle.

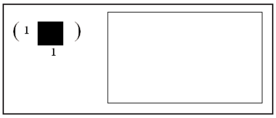


Figure 1 (the square centimeter is drawn on the E.G. worksheet only).

Task 2: Students were asked to measure the area of an irregular geometrical figure.

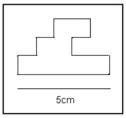


Figure 2

Task 3: Students were asked to measure the area of a 5cmx3cm rectangle using the formula length x height. Then, they were asked how many times the square centimeter fits into the rectangle.

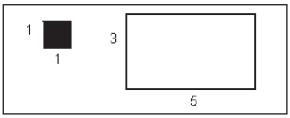


Figure 3

Task 4: Students were asked to measure the area of figures A and B using the given units a and b respectively. Then, they were asked whether either of the shapes A and B has a larger surface.

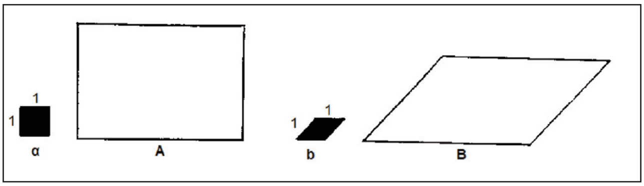


Figure 4