Constructing squares as a mathematical problem solving process in pre-school

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Abstract

Could problem solving be the object of teaching in early education? Could children's engagement in problem solving processes lead to skills and conceptual understanding development? Could appropriate teaching interventions scaffold children's efforts? The sample consisted of 25 children attending public pre-school in Cyprus. The children were asked to construct different sized squares. Findings show that children responded positively to the problem and were successful in solving it. During the problem solving process children demonstrated development of skills and conceptual understanding. Teacher-children and children-children interactions played an important role in the positive outcome of the activity.

Key Words

Problem solving, conceptual understanding, square, construction, pre-school education

Résumé

La solution de problème pourrait-elle faire l'objet de l'enseignement dans la première éducation ? Pourrait l'engagement d'enfants dans le problème en résolvant des processus cause des adresses et un développement de compréhension conceptuel ? Pourrait s'approprier des efforts d'enfants d'échafaudage d'interventions d'enseignement ? L'échantillon s'est composé de 25 enfants servant le public préscolaire en Chypre. On a demandé les enfants de construire de différents carrés de taille. Les résultats montrent que les enfants ont répondu de façon positive au problème et ont réussi dans la solution de cela. Pendant la résolution des problèmes *les enfants de processus ont démontré le développement des qualifications et de la compréhension conceptuelle. Les interactions entre le professeur et les enfants et les interactions parmi des enfants ont joué un rôle important dans les résultats positifs de l'activité.*

Mots-Clés

Résolution de problèmes, compréhension conceptuelle, carré, construction, l'éducation préscolaire

Introduction

Most pre-school education mathematics programmes tend to limit their aims to counting, recognizing and naming geometrical shapes, measuring geometrical sizes through direct comparison, and repeated patterns (Greenes, Ginsburg & Balfanz, 2004; Clements & Sarama, 2009). Nevertheless, current research shows that young children have the ability to develop informal forms of mathematical practices that demand the manipulation of complex mathematical ideas. Young children have also been noted to show an interest in the investigation and solution of mathematical problems (Greenes, Ginsburg & Balfanz, 2004).

Research findings in the area of pre-school mathematics education (Shiakalli & Zacharos, 2012), notes that young children (as young as four years-old) can acquire forms of mathematical knowledge that are more complex than was earlier believed. It is also argued that, young children are able to learn "more complex and sophisticated" (Greenes, Ginsburg & Balfanz, 2004, p. 160) mathematics than was suggested by pre-school mathematics programmes. Claims such as these about young children's abilities in mathematical understanding and mathematical problem solving are grounded in a solid evidence base (Ginsburg & Golbeck, 2004; Greenes, Ginsburg & Balfanz, 2004).

Findings also suggest that young children, have not only increased abilities, but also show readiness, enthusiasm and eagerness to be introduced to the world of mathematics (Shiakalli, 2013). Young children's interest in learning about the basic principles of mathematics is not significantly affected by "gender, socio-economic group, or ethnicity" (Greenes, Ginsburg & Balfanz, 2004, p. 160). Also interesting is the fact that young children's interest in mathematics is reflected in informal forms of learning, mostly during their free play. In brief, "virtually all children are capable of learning interesting mathematical ideas and methods" (Greenes, Ginsburg & Balfanz, 2004, p. 160) providing these are presented in a way the children will find attractive (Björklund, 2010; Zacharos, Antonopoulos & Ravanis, 2011; Zacharos & Kassara, 2012).

The present study is situated within the sphere of problem solving in general and more particularly within the sphere of mathematical problem solving. The problem children were asked to solve involved aspects of the properties of the square and counting. Moreover, in order to solve the problem children had to manipulate the concepts of geometrical angles and sides. Children were asked to construct different sized squares using specific material (six different sized squares could be constructed with the material which consisted of plastic rods). The teaching methodological approach presented here focuses on two themes: (1) mathematical problem solving, and (2) the development of geometrical conceptual understanding through the construction of squares.

Theoretical Framework

Mathematical problem solving processes

Children's pre-school experiences are believed to be critical, as they provide the child's first contact with the school environment and inform their perception of themselves as a learner. Therefore, pre-school programmes aim to support children in the development of effective learning strategies within the school environment where they develop an understanding of their role in the learning process.

Typically, the development and acquisition of the ability to solve mathematical problems is considered one of the basic aims of mathematical education for "doing mathematics in general, and solving mathematics problems in particular, is an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine" (Cai & Lester, 2005, p. 221).

Studying the processes of mathematical problem solving is not new to mathematics education research and there are numerous definitions concerning this matter. In the quest for a definition it is clear that it is difficult to find a single, unanimously accepted definition that reflects the complexity and richness of this process (Grugnetti & Jaquet, 2005).

Cai and Lester (2005) note that one of the factor that renders most definitions given for mathematical problem solving incomplete, is that most fail to clearly describe what the mathematical problem solving process is about. As a consequence, Cai and Lester (2005) chose the definition given by Lester and Kehle (2003) as the most appropriate: Successful problem solving in mathematics involves coordinating previous experiences, knowledge, *familiar representations* and patterns of inference, and intuition in an effort to generate *new representations* and related patterns of inference that resolve the tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity (Lester & Kehle, 2003).

A problem situation is defined as a new activity which is meaningful to the pupils and which must be sufficiently close to their current knowledge as to be assimilated and yet it must be sufficiently different in order to encourage them to transform their methods of thinking and working (Grugnetti & Jaquet, 2005, p. 375).

Leikin (2004) defined the characteristics of "quality" mathematical problem solving using a set of four conditions: first, the person who performs the task has to be motivated to find a solution; second, the person has to have no readily available procedures for finding a solution; third, the person has to make an attempt and persist to reach a solution; fourth, the task or situation have several solving approaches. Obviously, these criteria are relative and subjective with respect to a person's problem-solving expertise in a particular field, i.e., a task that is cognitively demanding for one person may be trivial (or, vice versa, unrewarding) for another (p. 209).

Leikin (2004) also introduces the idea that every mathematical problem should guide children towards new questions and new ideas, and underlines that these ideas should always be in agreement with the child's ability to solve mathematical problems as well as their familiarization and ability to use mathematical problem solving processes.

In studying different ways of mathematical problem solving, Deek, Turoff and Mc Hugh (1999) conclude that almost all mathematical problem solving processes follow four basic stages: problem formulation, solution planning, solution design and solution testing. Studying mathematical problem solving as a process Kappa (2001) adds one more stage: evaluation.

It is also argued that children's level of cognitive ability influences their ability to realize and set a mathematical problem. During this stage children activate and use their experiences and try to integrate new data into existing cognitive structures (Siegel & Borasi, 1992). Cognitive representation of elements comprising a problem challenges children's cognitive ability to link new elements into a pre-existing solid set.

Moreover, an important step in the mathematical problem solving process is children's ability to locate the relevant and important aspects of a problem as well as relevant and important aspects for its solution (Flavell, 1992).

Fortuno et al., (1991) note that children tend to stop the mathematical problem solving process when they find one solution. This is the reason why Kappa (2001) suggests children should be encouraged so that they continue to search for other solutions to a problem they have already solved.

Geometry in pre-school

Children begin formulating geometrical concepts much earlier than they enter pre-school. Moreover their ideas and beliefs concerning geometrical shapes (including incorrect ideas or misconceptions) become stable by the age of seven (Clements & Sarama, 2009).

Fuys and Liebov (1993) believe that the main aim of geometry programmes in pre-school should not be the teaching of the basic elements of typology nor to transmit knowledge concerning Euclidean geometry. Instead, pre-school programs should consist of geometrical activities designed to provide children with the opportunity to experiment, construct and develop geometrical shapes. The foundation of the development of mathematical thought begins from the ability to describe, use, manipulate, construct and deconstruct geometrical shapes. Through their natural human ability to transfer knowledge, children begin to construct and deconstruct numbers through the construction and deconstruction of geometrical shapes- children have been observed to transfer knowledge and skills from geometry to algebra (Reynolds & Wheatley, 1996; Clements et al., 1997).

Therefore, pre-school geometry programmers should be organized so that they consist of activities aiming at (Fuys & Liebov, 1993; Clements, 1999):

- eye-hand co-ordination
- developing the skills of observation, information interpretation and use of information,
- development of skills of comparison and relations discovery (similarities and differences) between the parts of shapes as well as between shapes,
- the development of the ability to verbalize thoughts and ideas,
- the development of the ability to discuss thoughts and ideas.

Under this light, the role of the teacher becomes increasingly important since she is called to use children's informal geometrical experiences to create the conditions for new ones so that children gradually move from random and informal solutions of geometrical mathematical problems to conscious and organized processes of formulating and solving geometrical problems. The role of the teacher is also important in creating such conditions as to give children the opportunity to talk about their experiences and discuss their thoughts and ideas with her as well as with the other children.

Our aim was to explore the possibility of pre-school children solving mathematical problems. The specific problem that the children were asked to solve was to construct different sized squares using specific material. The questions we set out to answer were:

- Can mathematical problem solving be the object of teaching in pre-school education?
- Can children's engagement in the problem solving process lead to the development of skills and conceptual understanding?
- Which is the communication frame between children and between children and teacher, during the teaching process, and how does it contribute to the development of skills and conceptual understanding?

The questions were explored through the creation of the construction of the square problem solutions.

Methodological Framework

Method

This study formed a part of abroade reducational program which extended throughout the school year. It included the development of educational activities aimed at the development of investigative learning practices, more specifically at mathematical problem solving practices. Each of the selected problems had to fulfill the criteria outlined by Arsac, Germain and Mante (1988): The pupil has to begin alone, new knowledge has to be constructed, the situation has to elicit a research process with repeated trials, conjectures and verifications, the situation must be self-correcting, the situation-problem must lead to the development of knowledge, anticipated by the teacher, which makes sense only if the situation-problem is shown to be necessary and efficient.

Moreover Leikin and Kawass (2005) note that every problem should guide children to new questions and new ideas. The problem presented in this article was related to the construction of the square and was presented to the children in January 2013.

The problem

Children were asked to construct squares of different sizes using plastic rods in six different lengths (each length was of a different colour) with the possibility of connecting to each other (Figure 1). By using this material the number of solutions to the problem was six. The children had to construct all six squares in order to have successfully solved the problem.

Figure 1

Material used for solving the mathematical problem

Participants

Twenty five children aged between four and a half and five and a half years old (mean 4.9) attending a pre-school public class in Cyprus participated in the study. The researcher visited the classroom daily and observed the activities relevant to the solution of the problem. All activities were carried out by the classroom teacher (in service public pre-school teacher with teaching experience of 13 years). The basic outline of the activities was designed by the researcher and classroom teacher after the former had informed the latter about the aims of the study and its basic theoretical framework. The teacher then proceeded in designing the actual classroom activities in detail.

The process

The children had no prior formal interaction with geometrical shapes or their properties. During circle time a hand puppet, Mr. Menelaos, visited the classroom telling children he had started playing with geometrical shapes and that he found one shape, the square, particularly interesting. Mr. Menelaos held a picture of a square and showed it to the class. He told the children that he wanted to construct as many different squares as possible using the plastic rods he had bought but he did not know how. Mr. Menelaos asked the children if they could help him and that he would return after a few days for the solution of the problem. The picture of the square was pinned on the classroom board (next to the "mathematics table") and the puppet left the room. The teacher discussed the problem with the children and presented the material that was to be used for solving it. The teacher explained that the material would be placed in the "mathematics learning centre"and children could use it in order to solve the problem during free activity time every day.

The Cyprus National Curriculum (Ministry of Education and Culture, 1996) states that from 7:45 a.m to 9:05 a.m daily, children must participate in "Free Activities". During free activity time children have the opportunity to engage in different activities (organized in "learning centres") aimed at the development of different social, cognitive and sensory skills. Children choose the "learning centres" they would like to work/play in. Some centres are designated by the National Curriculum (dollhouse, construction centre, art centre, language centre, drama activity centre) and can be found in all pre-school settings throughout the year, while others can be organized by the teacher depending on the children's interests or a specific curriculum topic and can be temporary.

During free activity time in November 2012, two months after the beginning of the school year, the classroom teacher organized a "mathematics learning centre" which consisted mainly of a round table seating four children at a time situated in the main area of the classroom. At first, free activity time spent at the mathematics table required the teacher to offer the children help and support. This was gradually reduced to enable her to observe and record the children's dialogue during the mathematical problem solving process. When working at the mathematics table the children engaged in solving problems with more than one solution. By January 2013 the children had already worked on a combinatorial problem with six solutions and a number analysis problem with seven solutions.

An observational grid adapted to meet the needs of the current study was used (Cohen, Manion & Morrison, 2004; Cai & Lester, 2005; Leikin & Kawass, 2005). The themes under consideration were:

1. Children's abilities in manipulating and understanding of the mathematical notions involved in the construction of the square problem solving process.
2. Children's conceptual understanding concerning properties of geometry.
3. The communicative forms developed during teaching. More specifically, we looked at the communication forms developed between children as well as between the teacher and the children. Concerning the teacher's communicative attitude we observed the following items: guiding pupils, giving pupils autonomy and alternating working modalities (individual work, group work, class work).

Data were collected through videotaped classroom episodes, the researcher's classroom observation records, the teacher's classroom observation records (after viewing the videotaped classroom episodes) and the teacher's reflective diary on classroom episodes. Initially, children worked individually while seated in a group of four. The teacher encouraged them to interact with the other children sitting with them and they were free to do so if and when they wanted.

It is important to note that the children were free to solve the creation of the square problem as many times as they wanted. They could return to the "mathematics centre" as often as they liked and chose to work on the specific problem either individually or in groups. The creation of the square remained the main mathematical problem of the "mathematics center" for one month (January to February 2013).

Materials

Children created their solutions using a plastic connecting rod set (Figure 1). This material enabled the creation of six different sized squares.

Data Analysis

In this section selected episodes of the mathematical problem solving process are presented. These episodes focuse on: (1) skillsinvolved in solving the construction of the square problem, (2) conceptual understanding deriving from the engagement with the mathematical problem solving process, and (3) communicative framework-forms of interaction- areas which were observed during the process, as stated above.

Skills involved in solving the construction of the square problem

While working on the construction of the square problem, the children had the opportunity to demonstrate a variety of mathematical skills such as counting, adding, subtraction, observation and comparison, strategy development and process organization. The following episodes demonstrate the previously mentioned skills.

Episode 1: Understanding the problem.

Charis (aged 4.7, pseudonyms are used for all children) was working on the problem for the first time. He had already constructed one square.

Charis: Now?

Teacher: What have you done here?

Charis: I made a square. I tried and now I know how.

Teacher: That is very good. Are you finished with the problem?

Charis: I do not know.

Teacher: Can you remember what the problem was?

Charis: Yes, to make different squares. Oh, I am not finished. I have to make more squares but different every time.

Teacher: That is correct. How are you going to make sure you do not construct the same shape every time?

Charis: I will put them all on the table. If I try with all colours I will have all the solutions.

Charis started creating squares using four rods of a different colour for each construction. Before taking four rods from the middle of the table, where the pile was placed, he looked at his construction in order to avoid creating a same construction. This enabled him to create all six solutions of the problem and place them next to each other on the table.

Charis demonstrated understanding of the request of the problem when after repeating it he realized that the solving process was not complete. When the teacher posed the problem of creating same constructions, Charis managed to think of an effective way to record his solutions: keeping them close to him on the table. By consulting his constructions, Charis managed to detect all six solutions of the problem without repeating any of them. Another interesting aspect of Chari's work was the ability he demonstrated in (1) organizing the process before acting upon it and (2) developing and applying an effective strategy.

In the following episode (Episode 2), Giannis demonstrated his ability to count and subtract and Pavlos demonstrated his ability to count and add while working together as a team on the construction of the square problem.

Episode 2: Children demonstrating counting, adding and subtraction abilities.

Giannis (aged 4.6) and Pavlos (aged 4.9) were working together as a team. Giannis had solved the problem once, working individually, while Pavlos had not worked on the problem before. Pavlos took seven rods (two red, three yellow, one purple and one blue) in order to construct a square.

Giannis: But these are too many. We only need four. Look at the picture, one, two, three, four *(Giannis touched the sides of the picture of the square placed on the board)*

Pavlos: Right, I took one, two, three, four, five, six, seven.

Giannis: Seven. That is three too many.

Pavlos placed three rods back and kept four in his hands (three in one hand and one in the other hand).

Pavlos: Now I am right. Three and one is four. Four rods. Let us try now.

Pavlos held seven rods. When counting them, Giannis observed that Pavlos had taken more than they needed. He continued explaining his thought ("look at the picture"). Without any difficulty Giannis was able to calculate the number of extra rods demonstrating the ability to subtract ("that is three too many"). Pavlos then demonstrated the ability to add when concluding that the total number of rods he held in both hands was four ("three and one is four").

We noted that frequently, children who had difficulties in manipulating the material were able to overcome their difficulties through their interaction with peers. During the development of the activity, we also noted that the role of the prototype and the imitator alternated, while the development of both "subjects" during the mathematical problem solving process showed that they were both cognitively working within their "zone of proximal development" (Vygotsky 1978).

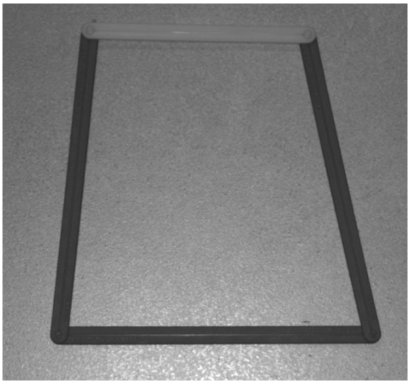
In Episode 2, Pavlo's interaction with Giannis helped him overcome the problem of having chosen more rods that needed for constructing a square. Pavlos had no difficulty in following Gianni's comments and actions. The two children seemed to had reached the same cognitive level. As a result, the less competent child (Pavlos) could benefit from the more competent child (Giannis) in understanding the process of beginning to construct a square.

Another important aspect of the episode recorded above is that Pavlo's final comment ("Now I am right. . . Four rods. Let us try now") as well as Gianni's initial comment ("But this is not right") showed that both children had developed epistemological readiness concerning the data and requests of a mathematical problem: in order to proceed to the investigation of the solutions of a mathematical problem the material used has to match the data of the problem.

In the following Episode (Episode 3) Maria demonstrated her ability to use observation and comparison skills in overcoming the difficulty she faced while trying to construct a square.

Episode 3: Children demonstrating observation and comparison skills.
Maria (aged 5) chose four rods- three red and one yellow. She connected the rods together forming a quadrilateral (Figure 2).
Teacher : Let us see. . . how are you doing?
Maria: I took four sticks because the shape has four lines. I connected them together but this is not the same as that *(pointed at the picture of the square)*.
Teacher: Why do you think it is different?
Maria: It is skew, that one is straight. It has a problem.
Teacher: What do you think the problem is?
Maria carefully examined her construction visually comparing it to the square in the picture on the board.
Maria: This piece has to go. I need another piece.
Teacher: Which piece are you going to chose to replace this one?
Maria touched the three equal rods.
Maria: I know, another red one. All four red ten.
In the above presented episode Maria's ability to critically compare her construction to the square she saw on the board, enabled her to correct her initial mistake and create a correct construction.

FIGURE 2



Unsuccessful attempt of constructing a square

Conceptual understanding
During the problem solving process the children had the opportunity to demonstrate conceptual understanding they had developed about the basic properties of the square (Episode 4).

Episode 4: Conceptual understanding of the properties of the square.
Costas (aged 5.3) had already solved the construction of the square problem twice

managing to detect all six solutions each time. He returned to the mathematics table six days later asking to work on the same problem again.

Costas: I know what to do. I need four same sticks every time.

Costas was observed to take four rods of each colour and place them in groups in front of him.

Costas: Now I have to connect them straight.

Teacher: What do you mean by "straight"?

Costas: They have to be connected like this *(took two rods and connected them in a right angle)*. Not like this *(closed the angle forming an acute angle)* nor like this *(opened the angle forming an obtuse angle)*. It has to be straight *(formed a right angle with the rods)*.

Teacher: Can you give me instructions to follow so that I can make a square?

Costas: First you take four same sticks *(the teacher took four purple rods)*, then you connect them *(the teacher connected them in a line)*. No, no close your line, bring the last and first together *(the teacher followed the instructions and created a rhombus)*. Be careful, this *(pointed at the angle he had previously created)* has to be straight always.

In the above Episode Costas demonstrated the development and formulation of conceptual understanding concerning the properties of (a) geometrical shapes: geometrical shapes are always closed and (b) squares: all sides are of equal length and all angles are right.

Apart from the development of conceptual understanding concerning the basic properties of the square, while working on the construction of the square problem, children demonstrated conceptual understanding they had developed about more general geometrical aspects such as parallels (Episode 5) and size (Episode 6).

Episode 5: Conceptual understanding of the notion of parallels.

Christina (aged 5.4) had worked on the problem three times. The teacher sat next to her once she had created all six solution of the problem.

Teacher: So have you found all solutions?

Christina: Yes.

Teacher: How would you help Mr. Menelaos with his problem?

Christina: I would tell him how to make squares. I know now. You have to take four same sticks and then connect them so the up and down sides are like this *(placed her hands in a horizontal parallel position)* and the two other sides are also like this *(placed her hands in a vertical parallel position)*.

Teacher: What do you mean like "this" *(placed her hands in a horizontal parallel position)*.

Christina: When the sides are like this *(took the teacher's hands and placed them* *horizontally parallel)* they will not touch anywhere. The same if they are like this *(took the teacher's hands and placed them vertically parallel)*.

Teacher: So can the lines not be like this? *(took a red and blue rod and placed them opposite but not parallel on the table. The two rods not touching)*

Christina: No because if you continue this line *(drew an extension of the top rod until she met the bottom rod)* the two lines meet. You see?

Through her interaction with the teacher, Christina had the opportunity to demonstrate her understanding of parallels. The most important aspect of her understanding was that she had understood that the length of the line plays no role in the phenomenon (the teacher took two rods of different length and placed them in a non parallel position) of parallels. What was important, and Christina showed to had understood, was the angle in which the lines were placed.

Episode 6: Understanding size.

Christos (aged 4.9) was working on the problem for the fourth time. He had constructed all solutions of the problem placing them in a row according to size. Stelios (aged 5.1) was sitting next to him still working on the problem. Stelios had worked on the problem two times before.

Christos: Look, they are all the same.

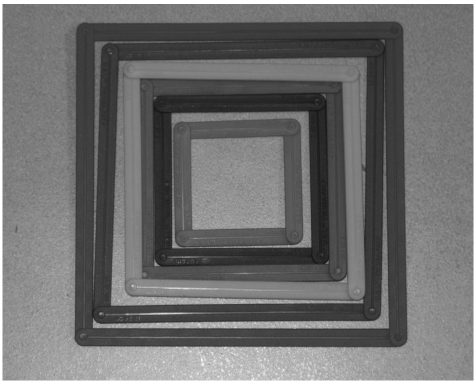
Stelios: They are not the same. They are bigger and smaller. They are all different.

Christos: No, same shape different size. Every time I change the size but the shape is always the same.

Stelios began to place each construction inside the immediate bigger one (Figure 3).

Stelios: Same shape different size, all the same.

FIGURE 3



Conceptual understanding: the size of the shape does not alter the type of the shape

Communicative framework-forms of interaction

Episode 2 recorded an interesting form of interaction between the two children. Firstly Giannis verbalized his observation stating his objection ("but these are too many"). He then moved on to give further explanation ("we only need four. Look at the picture, one, two, three, four"). Pavlos listened to what Giannis had to say silently agreeing (he placed the extra rods back). He then proceeded in confirming Pavlo's statement ("we only need four") by calculating the rods he was holding. Pavlos and Giannis demonstrated their ability to communicate effectively in order to overcome the problem they were facing (Pavlos having taken more rods than needed).

In Episode 3, like Pavlos and Giannis (in Episode 2), Maria also demonstrated her ability to communicate effectively. Whereas in Episode 2 the communication took place amongst peers in Episode 3 the child communicated with an adult (the teacher). As in Episode 2 verbal interaction led to the successful solution of the problem. More specifically, in Episode 3, Maria reacted to her teacher's comments and used the questions posed by the teacher in order to correct her mistake. On the other hand the teacher showed the ability to "scaffold" the child's thinking in a way that led to Maria creating a correct construction.

In Episode 4 the interaction between Costas and the teacher was of a different nature. The teacher acted as a less knowledgeable other giving the child the opportunity to demonstrate his conceptual understanding of the basic properties of the square and verbalize it.

In Episode 5 the teacher undertook a different role: through question posing she helped Christina to clearly express her thoughts.

Finally, in Episode 6, interaction between Christos and Stelios helped the latter correct his misconception about size effecting shape. As in Episode 2, both children seemed to be working within their "zone of proximal development".

During the initial phase of solving the construction of the square problem children chose to work individually- with limited interaction with the children sitting at the mathematics table at the same time. But during the course of the process they altered their attitude. Some children formed pairs. Some pairs created a kind of internal "differentiation" in their work: one child constructed a square using the rods while the other child observed the action. The roles changed for the construction of the next solution. Children were observed to intervene in each other's construction when they felt it was needed (Episode 2). It is interesting to note that children turned to their teacher mainly to announce an idea, or show a solution. Mostly, when faced with a difficulty, children would turn to another child rather than their teacher. Interactions between children (Episodes 2 and 6) and between teacher and children (Episodes 1, 3, 4 and 5) were particularly effective in the mathematical problem solving process.

We noted that frequently, children who had difficulties in manipulating the material in order to find the solutions would observe other (more able) children trying to imitate their work. By the end of the process the less able children were able to work entirely on their own (Episode 7). Moreover the interaction between "expert" children and "novice" children led to effective communication and conceptual understanding (Episode 6). Both above mentioned situations show that both children (prototype and imitator- expert and novice) were both cognitively working within their "zone of proximal development" (Vygotsky 1978).

Episode 7: Successful imitation

Zoe (aged 5.2) had already solved the problem three times. Eleni (aged 4.6) chose to work on the problem for the first time. Both children were working on the problem at the same time. Eleni had apparent difficulty in manipulating the material: she could not work out how to connect the rods. Eleni was observed to carefully study Zoe as she connected her rods. After watching Zoe connecting the rods in the construction of two solutions, Eleni took two rods and tried to connect them imitating Zoe. Her effort was successful. She repeated it a second time, being successful again. Eleni then started working on the problem trying to construct her first solution. She chose four different rods (yellow, blue, red, green). Seeing that her construction was unsuccessful she turned to Zoe.

Eleni: Mine is wrong. Why?

Zoe: Look at my ones.

Zoe picked up the square she had made using the yellow rods and gave it to Eleni. Eleni placed it in front of her and started creating her own square also using yellow rods.

Eleni: They always must be the same.

After having constructed her first solution Eleni returned the square to Zoe and continued her work until she managed to detect all solutions.

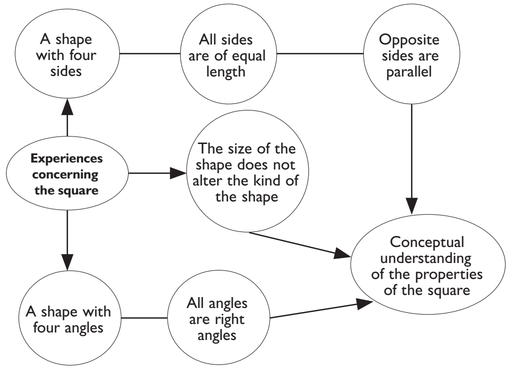
Conclusion and Discussion

In the present study we looked at pre-school children's ability to respond to mathematical problem solving processes such as the construction of squares. The children were required to construct six different sized squares using specific material.

Concerning the first research question- whether pre-school children can successfully solve the construction of a square problem and thus can mathematical problem solving be the object of teaching in pre-school education- all children responded positively. This was the result of different factors. Firstly, the gradual development of the problem during a period of one month which enabled children to systematically work on the problem led to the development of skills and conceptual understanding. Secondly, the teacher's contribution in (1) enabling and supporting the development of children's autonomy and (2) organizing the classroom in a way which gave the possibility for creative interaction to be developed.

During the mathematical problem solving process, children demonstrated numerous abilities such as the ability to count, add and subtract (Episode 2), the ability to develop geometrical conceptual understanding (Episodes 4, 5, 6) as well as their general ability to focus on a specific notion (Episodes 1, 3) and cooperate with their peers in order to solve a problem (Episodes 2, 7). A concise presentation of children's experiences leading to conceptual understanding concerning the basic properties of the square is shown in Figure 4.

FIGURE 4



Experiences leading to conceptual understanding of the properties of the square

Concerning the second research question referring to the ability of children to develop skills and conceptual understanding through their engagement in the mathematical problem solving process, findings suggest that children as young as 4.5 years of age can (1) develop skills such as counting, addition, subtraction, strategy development and effective communication and (2) develop conceptual understanding of notions such shape properties and geometrical relations (angle measurement, parallels) while applying the mathematical problem solving process.

This underlines the importance needed to be given by mathematics education to children's familiarization with the problem solving process as well as with geometry problems. The acquisition of skills relevant to the problem solving process during the early stages of education (as early as pre-school) can be especially useful to children since they, as students, will need to develop, use and assess skills and conceptual understanding acquired through the mathematical problem solving process in later stages of their school life (Ginsbur & Goldberg, 2004). Our findings also lead to the conclusion that the repeated creation of solutions led to conceptual understanding not only of the basic properties of the square but of broader geometrical notions as well (Episodes 4, 5, 6).

Concerning children's forms of interaction we observed several forms of collaboration between children which are consistent to the findings of other studies (e.g. Winsler, Diaz & Montero, 1997; Kieran, 2001).

Initially, cooperation among children was expressed through oral communication through which children helped each other by expressing their thoughts loudly. Another form of communication was developed when the child turned to another child's constructions. Finally when two or more children had difficulties in finding a solution the need for cooperation amongst them was created. In this case communication enclosed both above mentioned forms of communication. Cooperating children talked to each other and expressed their thoughts out loud, while at the same time they used their constructions to reach the desired result (Episodes 2, 7).

According to Davidson, Deasor and Sternberg (1994) children engage in mathematical problem solving organization when the process to be followed is new and the problem interesting. Research findings (Kappa, 1999) show that children'sproblem solving involves three basic strategies: (1) trial and error, (2) step-by-step planning and (3) holistic planning. In our study, most children initially tackled the construction of the square problem using the trial and error method. When they became more familiar with the problem, children used a step-by-step planning approach as well as holistic planning (Episode 1).

The role of the classroom context is substantial in the process of mathematical problem solving in pre-school. Similarly, geometry activities in pre-school education need to be embedded within a context which enables us to expect children's positive response. Such a context is functional for children's learning when it encourages children's interactions and also when the problem to be solved is within the children's "zone of proximal development". In these cases the children have the opportunity to develop practices of imitation of their more knowledgeable peers, as described by Vygotsky (1978): the subject's effort to reconstruct the action through imitation and the integration of its actions to its own cognitive and interpretative schemata. In the instances where the conditions set by Vygotsky (1978) were met, imitation was successful (Episode 7).

The teacher referred to abovewas not the exclusive centre of the process. The students' expectations and interests were transferred to the experimental atmosphere created by the teaching situation, scaffolding their autonomy (Episodes 1, 3) and providing multiple interaction experiences (Episodes 4, 5).

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