Pedagogy of research and questioning the world: teaching through research and study paths in secondary school

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Abstract

This paper summarizes some researchers developed in the Nucleus of Research in Education in Science and Technology (NIECyT) in the last eight years. In total, 17 implementations in mathematics classrooms where (N=234) students participated, are reported. The framework of the Anthropologic Theory of the Didactic (ATD) has been adopted, emphasizing the notions of Pedagogy of the Research and Questioning the World and the Research and Study paths (RSP). The results obtained introducing in a local and controlled way different RSP in regular courses of Mathematics in the secondary school in Argentina, are presented. The generative questions, the different mathematical and extra mathematical organizations, and the RSP are analyzed; also the dialectics and the attitudes of the Pedagogy of research and questioning the world are described.

Keywords

Anthropologic Theory of Didactic (ATD), Pedagogy of Research and questioning the World, Research and study paths (RSP), Mathematics Education, Secondary School

Résumé

Ce travail présente un résumé des recherches développées dans le Núcleo de Investigación en Educación en Ciencias y Tecnología (NIECyT) pendant les huit dernières années. Ont été réalisées, au total, 17 implémentations, dans lesquelles (N=234) étudiants ont participé. On utilise la Théorie anthropologique du didactique (TAD) comme cadre théorique, plus particulièrement, les notions de pédagogie de l'enquête et du questionnement du monde (PRQM) et les parcours d'étude et de recherche (PER). On présente une synthèse de quelques résultats obtenus après d'avoir introduit, d'une manière locale et contrôlée, différents PER dans des cours réguliers de Mathématiques à l'école secondaire en Argentine. On présente les questions génératrices et son arborescence, les différentes organisations mathématiques et extra-mathématiques que les parcours permettraient d'étudier, et les dialectiques développées dans des PRQM.

Mots-Clés

Théorie anthropologique du didactique (TAD), pédagogie de l'enquête et du questionnement du monde (PRQM), parcours d'étude et de recherche (PER), enseignement des mathématiques, école secondaire

Introduction

This paper summarizes various studies performed between 2005 and 2013 following the Anthropological Theory of Didactics, (ATD) (Chevallard, 1999, 2004, 2007, 2012, 2013). In total, a report of 17 implementations is presented, and (N=234) students have participated. The implementations tried to introduce into the typical classrooms of Mathematics in secondary school, the teaching by Research and Study Path (RSP).The RSP (Chevallard, 2004) consists of the study of Q questions as the starting point of mathematical knowledge, and the construction of possible answers to those questions produced by the class. In order to teach by means of RSP, a radically different pedagogical model, compared with traditional teaching, have to be performed.This new model is named *pedagogy of research and questioning the world.*

In the long term, the project promotes a reflective education, which allows further studies and life-long education. Each implementation of a RSP requires deep changes in the teaching of Mathematics in secondary schools, in this case, in Argentina. Therefore, teachers explain and promote feelings of admiration related to mathematical knowledge, like the products of mathematical cultural were pieces exposed into a museum. This phenomenon has been metaphorically called *monumentalism* (Chevallard, 2004). Within a monumental teaching, the autonomy and the activity of the students are absent, the students don't make decisions about what, how much and how to study; theyare only able to reproduce the knowledge in the same manner as it has been taught.

An unfortunate consequence of *monumentalism* is the absence of questions, which have been replaced by the teaching of answers. Knowledge, usually showed by the teacher, contains answers to "hidden" questions, for which the teacher does not make any reference related to their origin, usefulness, or even its *raisons d'être*. Thus, mathematics is taught as a finished and meaningless knowledge, which only has limited and stranger uses.

At school, the missing of the questions and of a genuine mathematical activity are some ones of the biggest difficulties in the reversal of current mathematics teaching. This is the problem we seek to address in the present work. Adopting the framework of the Anthropological Theory of Didactics (ATD) (Chevallard, 2013), this work attempted introducing the *pedagogy of research and questioning the world* (Chevallard, 2004, 2007, 2012, 2013) into secondary school teaching in a local and controlled manner. Thus, there is a substantial change to the prevailing pedagogical model, presenting an alternative to the *monumental pedagogy* and facing to the systematic elimination of the questions, all this realized through the Research and Study Paths (RSP) (Chevallard, 2009a).

Therefore, how can we introduce the RSP in classical and conventional lessons in secondary schools? How can the ecological problem be addressed, i.e., its adaptation or miss adaptation to the institution?, taking into account the importance of the necessary changes, the distribution of the obligations among the class members (*topogenesis*), the management of the time required regarding the time fixed by the institution (*chronogenesis*), and also, the balance between teaching aids and students' activities (*mesogenesis*). All these aspects can't be separately considered, because they will determine the scopes of the path.

Previous research on ATD has developed RSP in specially designed classes in college, which are far from the traditional university lessons (Barquero, 2009; Barquero, Bosch & Gascón, 2011; Ladage & Chevallard, 2011; Ruiz, Bosch & Gascón, 2005; Serrano, Bosch & Gascón, 2007). On the other hand, Fonseca & Casas (2009) analysed the transition from secondary school to college, developing a RSP that begins in secondary school and continues in the next year in college. The research experiences carried out by Fonseca, Pereira & Casas (2011), García et al. (2005) are set in secondary schools, the RSP were extra-curricular activities performed in special workshops.

In the opposite, the studies hereby reported were performed in conventional courses, in different contexts and institutions; hence, they might be useful in order to consider larger scale changes. We summarizes the results of three studies, whit distinct characteristics, like their features, the studied questions, all of them try to introduce relevant changes in the traditional teaching of Mathematics, by means of RSP.

Theoretical Framework: Main Results of Anthropological Theory of Didactics

Anthropological Theory of the Didactics (ATD) (Chevallard, 1999) developed the concept of *praxeology*, as a model to describe any human activity regularly performed. Praxeologies organize knowledge in two levels: *praxis*, linked to the know-how and types of tasks, problems and the techniques that are built and used to address them, and the *logos* or knowledge that corresponds to the descriptive aspects organized, as for example, the mathematical activity. Accordingly, the terms mathematical praxeological organisation, didactics praxeology, or just *mathematical organisation* (MO) and *didactic organization* (DO), will be used.

Didactics studies the conditions and restrictions of the institutional circulation of knowledge -praxeological-. The didactics occurs in the many social situations in which any person y or any institution *Y*, does intentionally something – or intends to do so – given that any person *x* or people *X* can study a work ♥. This works are any human material creations or not, deliberately produced in order to carry out a defined function (Chevallard, 2012). Didactics assumes the existence of didactic systems of type *S*(*X*;*Y*;*O)*, which operate according to certain rules, studied by Didactics.

Also, this didactic depends on the nature of the work ♥, the structure of the "*discipline*" to which it belongs (topics and questions) as well as on higher-level structures of the praxeological field that it comprises (sectors and domain). Although the didactic analysis initially consists in the praxeological analysis of the work ♥ it does not get exhausted by such an analysis (Chevallard, 2011).

The anthropological condition is formally expressed in the *scale levels of didactic codetermination*, which has been proposed and reformulated many times by Chevallard (2001, 2013):

Humanity ↔ Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline ↔
Domain ↔ Sector ↔ Theme ↔ Questions

In order to explain the reasons of institutional dissemination or the institutional non-dissemination of certain praxeologies, it is necessary to trace the origins of the mathematical knowledge (in the lower levels of the scale, as sector or domain) and at higher levels (pedagogy, school, society, civilization, humanity). Finally, the adjective "anthropological", point out that the domain of didactics is not restricted to the school, but that it may be extended to all social institutions where these works are disseminated, either if they are mathematical or not.

The didactic phenomenon of Monumentalism

The existence of the didactics takes on the recognition of facts or didactic phenomenon. The Anthropological Theory of the Didactics (Chevallard, 1999, 2007, 2009b,

2012) has largely identified and described the didactic phenomenon metaphorically named *monumentalism*, and its direct consequence: the loss of the purpose of mathematics at school. In monumental pedagogy (Chevallard, 2004), mathematics is taught in an atomized way, venerating the remains of monumental works, or monumentally claimed. Students are invited to admire these works, even though these lack utility and functionality. Another unfortunate consequence of this pedagogy is the implicit conviction of the futility of school mathematics, in order to face any problem in the real world, as well as the almost immediate oblivion of most of what has been studied

Within the ATD, alternative instruments to the dominant monumental pedagogy have been developed; which could allow the replacement of the traditional teaching model. The pedagogy of research and questioning the world is presented, as materialized in the RSP notion (Chevallard, 2001, 2009b). The RSP unfolds the need to redefine the school syllabus as a set of "generative" questions that allows discovering or re-discovering mathematic organizations proposed to be taught.

Research and Study Paths (RSP)

The Research and Study Paths (RSP) introduce a new epistemology, which returns the meaning and functionality to school mathematics (Chevallard, 2009b), while replacing the pedagogy of *"visiting works"*.

Students (X) research and study a question Q guided by a teacher (y) or a set of teachers (Y) with the aim of providing an answer *R* ♥ to Q. The exponent in *R* ♥ indicates that the answer to Q has been produced under certain restrictions, and it functions as a response to Q under such restrictions, as there is neither a universal nor a universally effective answer (Chevallard, 2009a).

The didactic system S requires tools, resources, works, i.e.it needs to generate a didactic environment M, to produce *R* ♥ (Chevallard, 2009a, 2009b).

$$[S(X;\ Y;\ Q) \rightarrow M] \hookrightarrow R^{\heartsuit}$$ (Herbartian Scheme)

The didactic system produces and organizes (➡) the medium M, that will allow to produce (➡) a response *R* ♥. The medium M contains the questions generated from Q, pre-built responses accepted by the school culture, noted as $R_i^{\lozenge}$ for i = 1, ..., n. Included within the category of "made $R_i^{\lozenge}$" responses could be a book, the Web, a teacher's course, etc. Also belonging to M, entities such as Oj, with j = n + 1, ..., m , such as works –theories, experimental works, praxeologies, etc.– can be found, considered potentially useful for the formulation of responses R and obtaining from it something useful to generate *R* ♥. In the ATD, M is written as follows:

$$M = \{R_1^{\lozenge}, R_2^{\lozenge}, R_3^{\lozenge}, \dots, R_n^{\lozenge}, Q_{n+1}, \dots, Q_m, O_{m+1}, \dots, O_p\}$$

A Research and Study Path (RSP) is represented by what Chevallard (2007) calls Herbartian Scheme, on behalf of the pedagogue Johann Friedrich Herbart (1776-1841).

$$[S(X,Y,Q) \rightharpoonup \{R_1^{\Diamond}, R_2^{\Diamond}, R_3^{\Diamond}, \dots, R_n^{\Diamond}, Q_{n+1}, \dots, Q_m, O_{m+1}, \dots, O_p\}] \hookrightarrow R^{\heartsuit}$$

At the beginning of RSP, the teacher presents a question *Q*, which study will lead to encounter various mathematical organizations and / or other disciplines, depending on whether it is a monodisciplinary or co-disciplinary path. Accordingly, a chain of questions and answers that are the core of the study process is established $P=(Q_i{:}R_i)_{1\leq i\leq n}$ being Q_i all the questions that live in the heart ♥ and R_i their respective responses (Chevallard, 2007).

Developing an RSP, radically changes the relationship between teacher and student in regard to knowledge, the time and the way in which the study is organized and the place that the members of the didactic system take in the class, according to the description below.

Topogenesis, chronogenesis and mesogenesis

The processes called *chronogenesis, mesogenesis* and *topogenesis* are *didactic functions* or *production functions* (Chevallard, 2009b) and they have a long tradition in the ATD, considering that they are already referred to in the epilogue of the book about didactic transposition.

Mesogenesis is the building of M, using both external and internal productions. The latter include responses R_x proposed by pupils x, being the result of their mathematical activity. Various types of works can be, in principle, included within M in a RSP; works that in traditional teaching would be excluded by principle. As the nature of the environment is modified, the "work" carried out is also modified. The mesogenetic conditions are related to the topogenetic ones, *M* is a production of the class [X, *y*] and not only teachers' *y* product.

The students' *topos expands* as, in addition to *their* personal response R_x, they may also introduce into M any works they decide to include. The teacher's topos is also significantly affected; the teacher may introduce certain institutional response or cultural $R^{\Diamond}$ into *M*, which will not necessarily be "their" personal response R_y. The teacher loses centrality during class, and becomes another *media*, provided that no *media* will have the privilege of *"being believed because someone said so"* (Chevallard, 2009a).

In RSP, the chronogenesis is affected by the intrinsic gestures of the study, related to the need to generate M and to produce a response. This involves a considerable expansion of the didactic time, i.e. an extension of the clock time required (Chevallard, 2009b).

The activities carried out in *M* building a response *R*♥ entaila *roughly* induced study of works O_j and a momentary transformation of *S(X;Y;Q)* in a "classical" didactic system $S(X;Y;O_j)$. The work *Oj* is studied for the purpose of the development of *R*♥, and it may get expanded into other works O_k, as far as the investigation requires it, and not because *y* has to introduce it or has artificially induced it.

Fundamental dialectics of a Research and Study Path

According to Chevallard (2001, 2007, 2009a, 2013) in any RSP there are certain didactic gestures characteristic of the study and research that must be done, called *dialectics* or ways of knowing-how to. The number and description of the dialectics has varied over time, as provided by the results from the research on the ATD. At present, they are the following:

- Dialectics of study and research: it is the heart of an RSP teaching and of the pedagogy of research and questioning the world. It suggests that it is impossible to do research without studying, as well as that genuine study will generate questions to be researched on.
- Dialectics of the individual and of the group: in the RSP the questions should be studied in groups. Sharing responsibilities is vital and assigning more or less individual tasks, so as to put them together in the final task of developing an answer as group work. The responsibility of the study does not lie in the individual but in the classroom community, which supports and validates the responses it generates.
- Dialectics of analysis and synthesis, praxeological and didactics: every didactic analysis entails a *praxeological analysis and backwards*. In order to understand a mathematical organisation (practical, technical, technological or theoretical) it is necessary to set didactic questions and to analyse: Where does this praxeology come from? How has it come to be learned by the institution in which it has been discovered? How did it get to this stage and which changes have it undergone? Besides, a didactic analysis requires considering: what is the praxeology we intend to teach like?
- Dialectics of getting into and out of the subject: if the question is broad and generative, it is necessary to provide the possibility of "getting out"of the subject and it may even be necessary to leave the discipline of reference, and go back to it later. In order that any generative question originates new extensivere search and study paths, it should involve more than one area and more than one discipline (Chevallard, 2007).
- Dialectics of parachutist and truffles: it is inspired by the words of the French historian Emmanuel Leroy - Ladurie, who classified historians into parachutists and truffle hunters. The first ones explore large areas of territory, inspecting panoramically the entire field. But they perform this exploration from so high that they fail to see any details. Meanwhile, truffle hunters[1] bring to light buried treasures and scrutinize the surrounding ground with their *nose on the ground*. This dialectics gets opposed to the school habit of finding instant solutions, thus transforming the process into something fast and easy, where responses are presented as immediate, and "remaining " for only one class.

1 Truffles: type of fungus and a coveted delicacy from ancient ages to our current days.

- Dialectics of black boxes and clear boxes: it refers to the need to decide if a work deserves to be studied, clarified, analysed, etc., or whether certain knowledge will be left in a *grey level* (Chevallard, 2007). The knowledge that is not essential to answer the generative question or its derivatives could be left in grey. This position is opposed to the school habit of expectingan absolute clarity, an exhaustive and specialized study, as if knowledge could somehow be depleted.
- Dialectics of the environment and the media: refers to the fact that the response $R^{\lozenge}$ obtained by RSP has to be tested. A "media" is any communication system (a newspaper, a television program, a teacher´s course, an "erudite" treaty, etc.). In order to develop possible responses, students count onsome already established answers acquired through the media. Then, these answers will be put under trial and transformed so that they can be incorporated into the environment.
- Dialectics of reading and writing: it refers to the avoidance of the transcription of already existing partial responses which are considered as relevant –while there is a questioning of the text where they were found– and the only extraction of the useful parts to be rewritten in notes, summary notes, etc (Chevallard, 2007). This dialectics is connected to the black and clear boxes one, as it is necessary to establish the depth level in which the existing partial responses were transcribed.
- Dialectics of production and reception: it refers to the need to spread and uphold the developed response. Knowledge by its own is not important –monumentalism– but it becomes relevant because the mathematical activity provides valuable answers for the community of study.

The theoretical constructs developed in this section, have been taken over by the different studies in order to analyze the results which were obtained in each case.

The following sections briefly describe the methodology used in the studies presented in this paper and some of the outcomes of the RSP implementations performed in the last eight years. Firstly, the team carried out paths restricted to exclusively mathematical questions, which led to the longitudinal study presented hereby. Then, some paths involving mathematics and other disciplines, i.e. microeconomics or physics,were proposed; being only the first case presented here, and finally we've done implementations of the same path into three other institutions. The results provided by these three studies are summarised and their characteristics are described in detail in the following section.

Research Methodology

The research is longitudinal exploratory, qualitative and ethnographic. The paths are described as well as their set up in conventional secondary school lessons is analyzed.

In all these studies, the implementations were directed by the researchers. The three studies summarized in this paper were developed as follows:

- Study 1 (S1) develops de results of a mono disciplinary RSP. The results achieved are exclusively mathematical. The singularity of this studyis that it is a longitudinal research performed during two years with the same group of students. In total, N=163 students were part of the investigation, carried out in 12 implementations, i.e., three studies of cohorts of two years each, in two parallel courses each year, always in the same institution.
- Study 2 (S2) corresponds to a Microeconomics bi-disciplinary RSP, which was developed during eight months in conventional Mathematics lessons in the last year of Secondary School. 28 students have been participated in this RSP.
- Study 3 (S3) is an Economy bi-disciplinary path which was particularly carried out in three different institutions (I), deliberately selectedby the researchers: I1 is a private management state school, I2 is a state school for adults and I3 is a private school. In total, 4 implementations were done (2 in I1, 1 in I2 and 1 in I3) considering all the institutions. Each intervention was realized during three months of fieldwork for each institution. In total, N=101 students have been participated. It is important to consider that all the students in Institution 1 (N=58), had previously participated of S1.

In all the studies mentioned, the written protocols of each student in all the lessons are obtained. After, these protocols were digitalized and were taken back on the following lesson. Audio recordings of the lessons and field notes were carried out. In all cases, the teachers were the researchers and the teachers from the team were in charge of the courses, except for I2 from study 3 which corresponds to a school for adults, in which the teacher in charge was not part of the team, but who handed over his/her course to the researcher, who took over the role of teacher henceforth. Every meeting is recorded in a "general" audio and field notes are taken by participant and non-participant observation. From all these studies, the protocols and results of N=234 students were obtained. The singularities of each study (S1, S2 and S3) are described as follows.

Study and Research Paths in Secondary Schools

The studies described in this part of the work have been performed in conventional courses in secondary schools. In the first study (S1), the characteristics of a mono disciplinary Mathematics RSP are analyzed, which allowed the development of a cohort study in two consecutive years. The second study (S2) presents the results of a bi-disciplinary RSP in Mathematics and Economy, and the results are relative to the role of the dialectics, as the students' protocols have shown. The last study (S3) describes the results of a bi-disciplinary RSP also in Mathematics and Economy but implemented in three different institutions, being the main results relative to the attitudes that are inherent in the Pedagogy of Researchand Questioning the World and its scopes in every Institution. As follows, the results obtained in each study are provided.

SI: Monodisciplinary Research and Study Path in secondary school

The first study involved the design, implementation and evaluation of a monodisciplinary RSP (Chevallard, 2009b), here, the generative question is specifically mathematical. It started with Q_0: *How to operate with any curves, knowing only its graphical representation and the unitof the axes?* (Llanos & Otero, 2013a, 2013b). Possible answers to Q_0 generate potential paths, according to the questions arising from Q_0, which is called Q_i. They allow a relatively complete coverage of the Mathematics curriculum in the last three years in secondary schools.

Question Q_0: *How to operate with any curves, knowing only its graphical representation and the unit of the axes?* It was adapted from a problem proposed by Douady (1999) to study the signs of polynomial functions. According to the type of curve and possible operations between them, different MOs could be obtained. In Scheme 1, the MO of the Mathematics program of secondary schools that could be studied is presented, as well as some arising questions, which were obtained as a result of the implementation of Q_0.

The RSP was developed in secondary school Mathematics lessons, with students attending 4th and 5th year, in which the teachers were the researchers. Its application started in the 4th year and continued in the 5th year, with the same group of 14-17 year-old students. Once Q_0 was introduced, the initial responses of the students determined the possible paths, depending on the previous knowledge of the pupils. The MOs effectively built during the RSP, have been noted in Scheme 1 with the dashed lines.

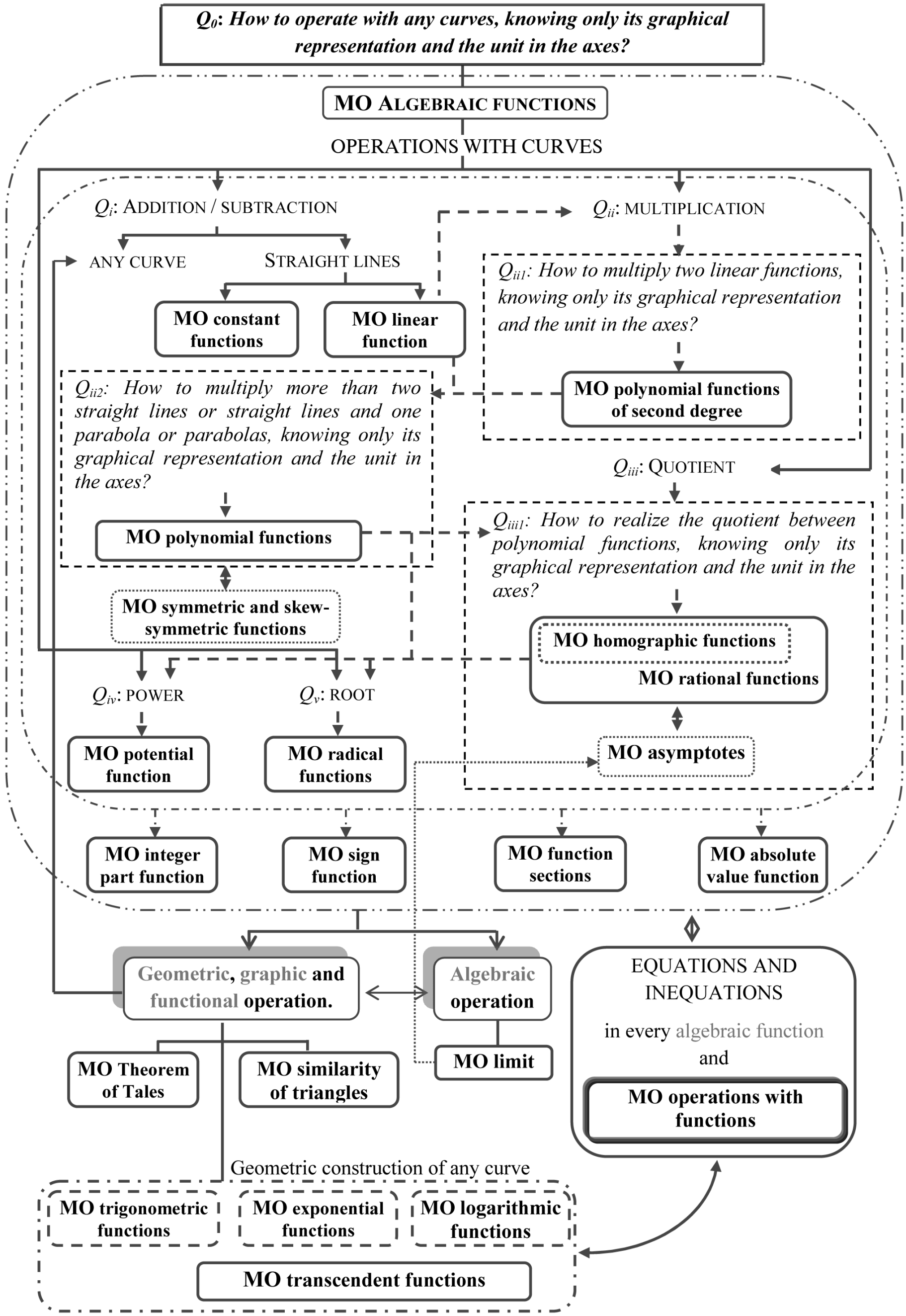
The students suggested the study of the multiplication of two straight lines. They obtained the MO polynomial functions of the second degree, which are part of the 4th year curriculum in secondary school. Subsequently, they studied the MO of polynomial functions of second degree, the MO of polynomial functions and the MO of rational functions, in three parts (P_1, P_2 and P_3).

P1: Relative Notions on the MO of polynomial functions of the second degree.

In order to answer Q_1: *How to multiply two related functions, knowing only its graphical representation and the unit of the axes?* Variants of the problem were proposed and a graphic representation of parabola $h = f \cdot g$ is showed in the following situations:

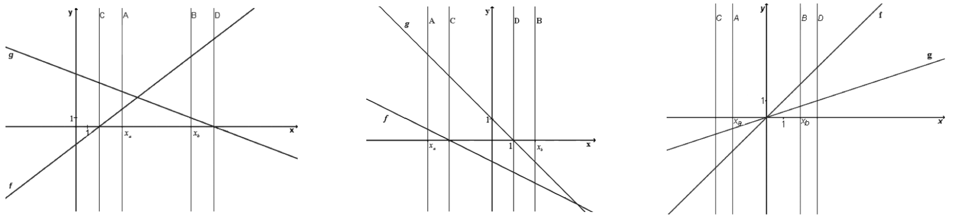
Functions f and g are presented in the graphic representation. All straight lines A//B//C//D, are perpendicular to the axis x. Function h = f g

Scheme 1



Description of potential MOs that allow studying the RSP (Llanos & Otero 2013c, Otero et al., 2013)

Figure 1



Graphics corresponding to functions f and g

a) Which is the most reasonable graphic for h?

b) What characteristics from graphic h could you justify? For every x_a y x_b equidistant from the zeroes in each function $\overline{CA} = \overline{BD}$. Is it correct to say that $h(x_a) = h(x_b)$?

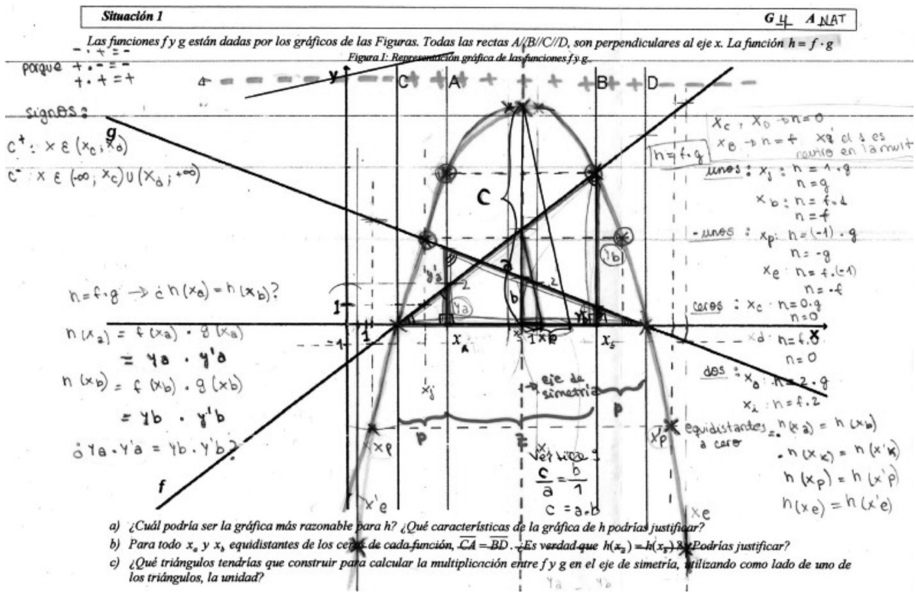
c) Could you justify it? Which triangles need to be constructed in order to calculate the multiplication between f and g in the symmetry axis, providing the unit is the side of one of the triangles?

It is necessary to identify the outstanding points, test the symmetry of the curve and build the apex, using the only available resource: geometry. With the purpose of testing the symmetry of the curve, it is necessary to construct triangles that are equidistant from the zeroes by using the segments determined bythe straight lines in those points, showing that they are similar and using the Tales' Theorem, to establish the geometric proportion between the segments determined by these triangles.This allows determining the axis of symmetry as well as the symmetric points. So as to determine the apex it is also necessary to construct similar triangles, using as information the unit of the abscissa axis. A detailed analysis of how to perform geometric calculations can be found in Llanos & Otero (2013a).The protocol of Figure 2 illustrates the type of responses which were obtained.

Subsequently, the teacher presents new questions arising from Q_0 in the analytical framework. Starting with the product between straight lines; analytical representations for the second-degree polynomial functions resulting from a factorized expression were obtained. In the analytical framework the zeroes and their properties, the multiplicity of the roots, the maximum or minimum, and the signs of the functions are reinterpreted-first geometrically and then analytically built.

Taking into account the case of the imaginary roots, it is necessary to go back to the geometric frame to analyse the translation of vector $\vec{u}$ from a given graphic. The techniques developed by students in response to the problem of multiplication of two straight lines in PI were generalized and adapted to the multiplication of more than two straight lines, or straight lines and parabolas or between parables and quotient.

FIGURE 2



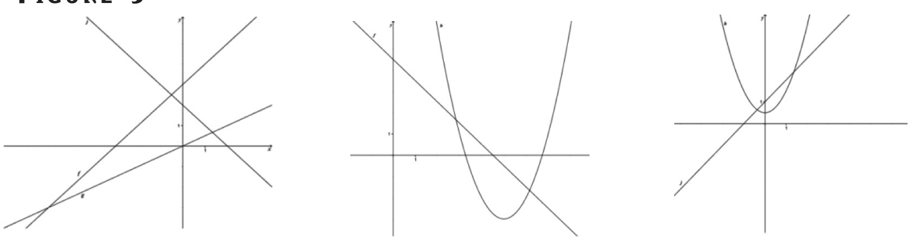
Protocol corresponding to student A161

P_2: The MOs of polynomial functions in the RSP

Question Q_2 is: *How to multiply more than two straight lines, or straight lines and parables, or parables, if only the graphs and the axis unit are known?* The proposed situations were:

Functions f g y j o f y h, are given by the graphic of Figure 3. Function p = f · g · j or = f · h.

FIGURE 3



Graphics corresponding to functions f, g and h respectively.

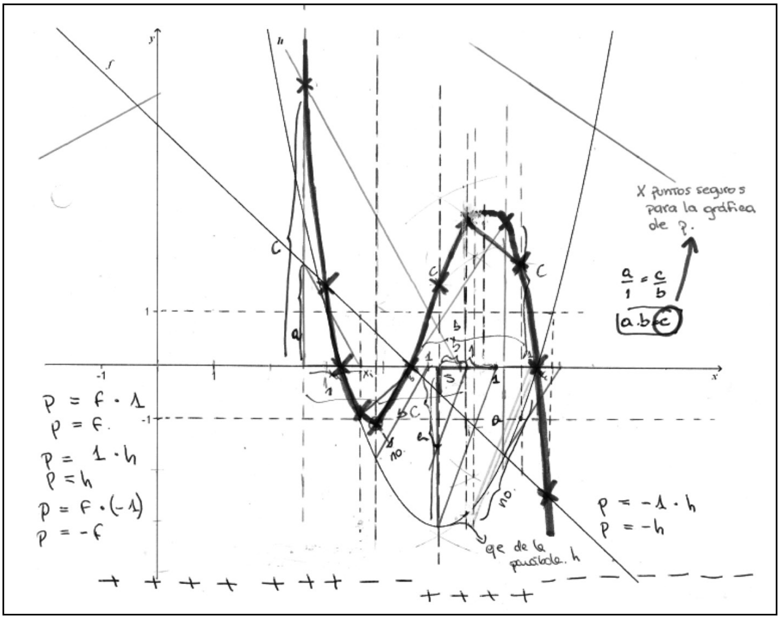
a) Which are the "safe points" and the signs of p?

b) Which could be the most reasonable graphic for p? What features ofthe graphic of p could you justify?

Then, students adapted the techniques generated for the previous case and they focus on the *safe points*: zeroes, ones, minus one, multiples of the unit; and they previously analyse the signs (C^+ and C^-) which the product may have when they attempt to get the curve for *p*.

Student AI28 protocol presented in Figure 4 shows how to get the curve of a third degree polynomial function when multiplying three straight lines, only when it's graphical representation and the axis unit are known.The analytical representations of polynomial functions were studied, first as multiplication of curves and then, the general form of these functions was obtained analytically. Finally, the students developed techniques to perform operations with polynomials –algebraic and graphically– studying the equations associated with these functions.

Figure 4



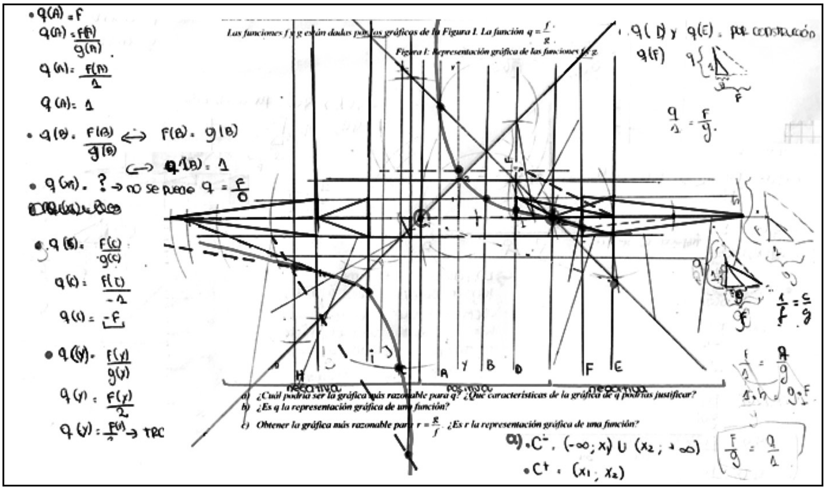
Protocol corresponding to student A128

The construction of the curve that results from multiplying other curves with a lower grade, gives meaning to the factorization of polynomials, as well as to the meaning of zeroes and the factorized form of these functions – aspects that are reconsidered in the last part of the path (P3). By analysing the quotient between polynomial functions, the rational functions are found and the techniques previously generated are readjusted.

P_3: The MOs of rational functions in the SRP

When responding to Q_3: *How to get the quotient of polynomial functions, if only its-graphical representation and the axis unit is known?* Students readjust the geometric techniques for any abscissa (Otero, Llanos & Gazzola, 2012). As an example of the type of tasks performed, protocol A81 is proposed.

FIGURE 5



Protocol corresponding to student A81

Rational functions were also studied in the analytical framework. It seems that by starting with factored expressions, it may have contributed to the results obtained at the end of the path, and especially, in regards to the work performed with simplification techniques, and the operations with rational functions and their properties.

The empirical foundation built, which results from the 12 implementations performed, allows to statethat finished RSPsare a suitable tool to start ecologically viable changes in the teaching insecondary school level. However, they showed limitations related to a complete experience of the pedagogy of research, particularly in relation to the dialectics. Thus, a co-disciplinary path development was attempted, focusing the analysis on the characteristics of the dialectical involved in an RSP.

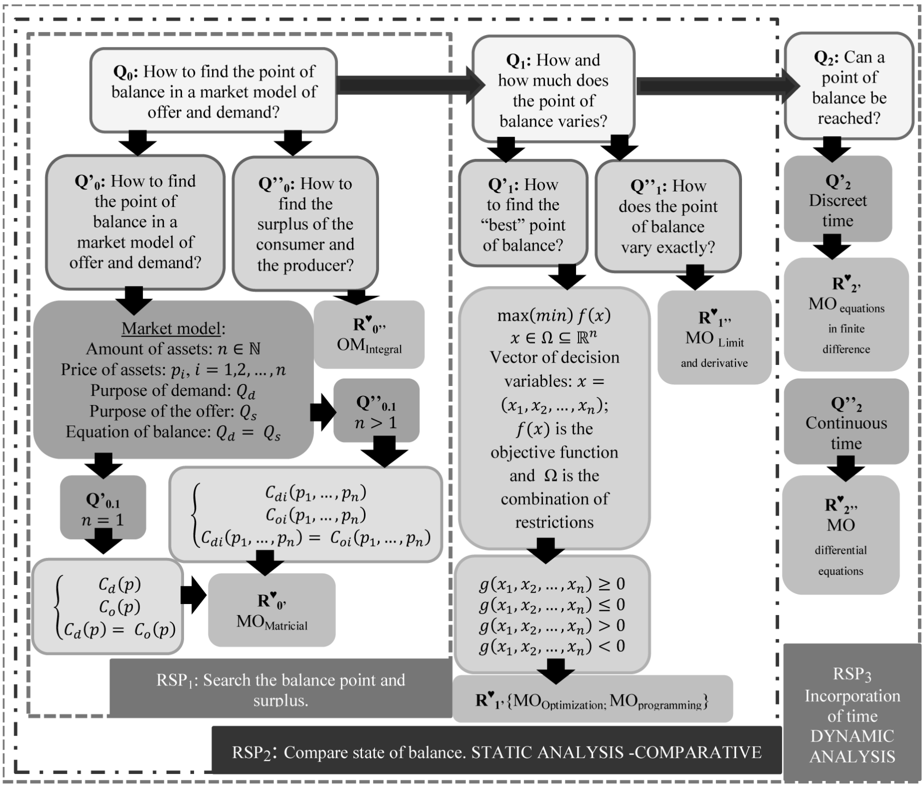
S2: Co-disciplinary RSPdevelopedin secondary school

At this stage of the project a RSP was implemented, analysed and evaluated during the last year of secondary school. The generative questions were provided from Microeconomics and the study process was analysed from the functioning of each one of the dialectics.The students started with the following questions: How do we find the break-even point in a market model? How and how much does break-even pointchange if the parameters in the model are varied?

The research was performed in a last year course of secondary school (students aged 17-18) from the very first day of the school duringeight months.

The praxeological model of reference (PMR) allows the analysis of possible paths and its possibilities, depending on the types of problems and questions related to the break-even point between supply and demand in a market. Possible RSPs depending on the hypotheses and questions are summarized in Scheme 2.

SCHEME 2



Description of possible MO that allows the study of co-disciplinary RSP (Parra, 2013)

It was decided to analyse the variations of a linear model of supply and demand, where the equations depend only on the price of the goods (p) and the quantities of it (being Q_d the quantity demanded and Q_s the quantity supplied). The teacher introduced the questions to which each group of students had to provide an answer; and if necessary, to suggest new questions, so that then they could be communicated and supported in front of the rest of the class. The classroom (study community) decided which questions to answer among those generated by the groups.

The RSP generative questions were formulated as follows:

Q_1: Imagine you are producing stuffed pasta to sell and raise money. From a prior sales study the following information has been obtained:

Quantity supplied (in dozens)	Price per dozen (in $)	Quantity demanded (in dozens)	Price per dozen (in $)
155	10	330	7
307	18	250	15
98	7	270	13

How can we decide at what price per dozen all that is produced is sold so that there is no unsatisfied demand? What linear model would study the behaviour of the supply and demand of this market?

Q2: How would you study the behaviour of the laws of supply and demand for any pair of linear functions of supply and demand? How would you find the break-even point in this case?

Q3: Imagine that the supply function of a market model is given by the function $O(p) = 3p - 2$ and that the demand function $D(p) = -4p + 6$. How would you describe the variation of the break-even point byonly modifying the value of the ordinate of demand? What if instead we change the value of the ordinate of the offer?

Q4: Considering the same functions of supply and demand in the previous case, how would you describe the variation of the break-even point now byonly modifying the value of the slope of the demand function? What if we only modify the value of the slope of the supply function?

Q5: So far we have been able to determine the way in which the price and the quantity will vary according to the slopes' variations of the supply and demand functions. But how much does the break-even point exactly vary in each case?

This particular path had allowed to rediscover the MO relative to the *linear function*, the MO relative to *straight lines in the plane,* the MO of the *systems of two linear equations and two variable*s, and the MO relative to *the limit and derived from function*s, from the variability perspective.

Some results on the functioning of the Dialectics

The functioning of the dialectics allows us to describe the dynamics and ecology of RSP, since they operate repeatedly throughout all their development (Parra, Otero & Fanaro, 2013a, 2013b). Regarding the dialectics of *"study and research"*, in order to answer the questions, it is necessary to study Microeconomics and Mathematics and researching into both disciplines. The study and research on these questions demands a decision on the media in which it is possible to look for those answers (Internet, Microeconomics books, Economics, Maths books, teachers of economics, and teachers of Mathematics, among others). After performing the research, it is necessary to determine which *media* would be incorporated into the environment (dialectics of environment -media) and what kind of knowledge extracted from them will be tested, rather than accepting them without questioning. The searching in different sources of information requires determining what knowledge is relevant and useful to build the response *R*♥, and how deep they have to be studied: "black boxes and clear boxes". This leads to decide how such knowledge will be included in the construction of *R*♥, being necessary to analyse, interpret and rewrite such knowledge rather than copying it word by word (reading and writing dialectics). On the other hand, *R*♥ has to be discussed and agreed by all of the members of each group. Although an individual study and research could realized, at some point they must contribute to the collective response (individual and collective dialectics). Once each group has constructed *R*♥, they communicate it to the rest of the class (dialectics of dissemination and reception).

All the above actions were performed "out" and/or "entering" in various mathematical and/or microeconomic works at different times of the study. For instance, students knew the limit of the functions in one variable, but they had to study it again, if the desired response needed so. This caused a "going out" to the limit of functions, which lasted over time, but which gave meaning to the "limit of functions.

The dialectics of the didactic analysis (and synthesis) and the praxeological analysis (synthesis) are carried out both during the RSP development and in its didactic engineering. During the design process and implementation of the RSP, the researcher analyses the didactical and praxeological actions that each question can bring into play. This analysis is materialised in the recurrent construction of the PMR.

All mentioned dialectics could be interconnected and each one's functioning could activate the others.

So far, during the entire path, a critical aspect is connected to the difficulties of the teacher to take their place, without yielding to the pressure from their students to "provide them the solution". This limits the possibilities of teaching based on the pedagogy of research and questioning the world in case the teacher is not familiarized with ATD, or with the pedagogy of research or if the mathematical background is weak and is not supported by a group; in contrast with these cases where the teachers are the researchers of the team.

After eight months of fieldwork, several questions remained unsolved. How can we cope with the demands of students and reduce the anxiety and uncertainty generated by the new responsibilities? How can we gradually bring forth the autonomy necessary for students to abandon the habits of obedience and dependence created by the traditional pedagogy? How can we carry out the qualification process when applying RSP? What know how teachers need in order to teaching by RSP?

S3: Co-disciplinary RSP in three secondary schools

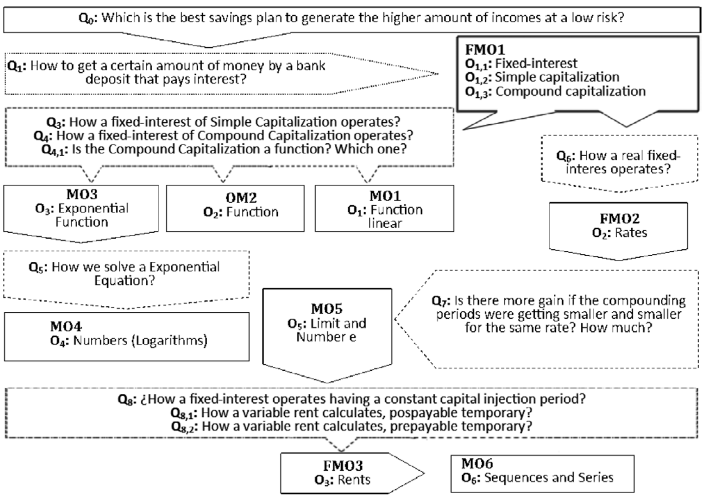
In the third part, the analysis of the ecology of an SRP based on the same generative question in three different school contexts was carried out. The questions were: How would the institutions affect the ecology of the RSP? What praxeologies could be found in each of them?

The same generative question was formulated in the three institutions (Donvito, Sureda & Otero, 2013). A developed PMR scheme is presented in Scheme 3. The RSP would allow to find three financial mathematic organizations (FMO1, FMO2, FMO3), three relative to the study of functions (MO1, MO2, MO3), one concerned with the study of numbers and their properties (MO4), another one on the study of sequences and series (MO6), and a third one linked to the study of the limit (MO5) .

Institutions (I1, I2, I3) are characterized as follows:

Institution 1 (I1) is a public secondary school with private management. The RSP was

SCHEME 3



Description of possible MO that allows studying the RSP implemented in two fifth-year courses, with a total of 58 students (aged between 16-17). They had two years' experience in the *Pedagogy of Research and Questioning the World*. The students had computers and internet at home.

Institution 2 (I2) is a private school where students are required to carry devices such as netbooks or tablets, as they have internet access during class. This case involved 25 students from fifth year.

Institution 3 (I3) is a public school for adults. The group is very heterogeneous and is composed of 18 students aged between 16 and 60. Most of them have various jobs and have resumed their studies (after 1-30 years). In all of the three schools, students formed groups of 4 or 5 members each. Only students I1 have had experience in the new pedagogy.

Institution 1: In I1, the students collected the money for paying their graduation party. They paid a monthly fee and placed the amount collected in fixed-term deposit, adding every month the total amount collected. Taking advantage of this situation, an RSP was presentedby the question Q_0: Which is the best savings plan that generates the most revenue, with lower risk?

To understand Q_0, students suggested new questions:

What is a fixed term? How are interests paid? What is the bank benefit? Which bank pays more interest? What currency can be or is more convenient to use in the deposit? Is there a mathematical formula that calculates given procedure? What is the gain of the fixed term depending on the amount and timing? Among others.

The students studied about fixed terms, the laws of simple and compound capitalization and nominal rates, effective and equivalent rates. Mathematical aspects of capitalization led to study the MO3 relative to the Exponential Functions and Exponential Equation. Also, a text prepared by the teacher based on the exponential variation, the vertical displacement of the exponential function (EF), analysis parameters, exponential equation, zeroes of the EF and the EF analysis, were used.

Most groups calculated recursively the savings plan. With the purpose of avoiding the recursive calculation, they sought a formula to calculate the money they would have each month, including the aggregate share. This led them to questioning *how to calculate a fixed term that has a periodical addition of money?* It was necessary to go *into the issue* of Series and Successions (MO6) to study income, successions and geometric series, and then *leave it and return* to the main question. Some groups designed savings plan combining fix terms in a spreadsheet and even obtained a suitable formula.

Summarizing, in I1, it was possible to develop some of the pedagogy of research gestures, and rebuild many of the organizations referred to in Scheme 3.

Institution 2: In I2, students expressed dissatisfaction because they were not used to study questions such as Q_0. The teacher had to suggest the questions: How can you capitalize in order to get a million AR$? What do we mean by putting money at interest? What is a fixed term? Etc.

Students immediately resourced to the internet, but their searches were short and quick, and their answers were brief and incomplete. With an algebraic expression for compound interest that they obtained on the web, and which the whole class accepted without questioning, they tried to get a million AR$ with an annual rate of 45 % and an initial capital of about AR$ 800,000. They did not use internet to know the market rates, and they solved the task ignoring the benefits they could obtain from that information. They concluded that "*so as to get one million pesos, we need a lot of initial capital (say AR$ 800,000) and / or a long time*" and then they abandoned the investigation. The teacher asked them to state explicitly the type of capitalization used, and students associated the Capitalization to the Linear Function, and the Compound Capitalization to the Quadratic Function, saying that the graphic was a "*half parabola*". The teacher asked about this statement and, in order to justify it, the students transcribed geometric definitions of the parabola from the internet, later failing when they were asked to provide an explanation for it.

In short, the pedagogy installed at this institution made the RSP unviable.

<u>Institution 3</u>: In I3, the teacher in charge of the course, asked to replace Q_0 by the following: What is the best savings plan to generate a million dollars with low risk? Putting forward the students'inability.

The groups could only produce closed, shallow, or dichotomous questions: "To a bigger amount of money, is the interest higher?", "Are interests variable?", "How long should the money deposit be? 30, 60, 90 days long?", "How much money should we deposit?","What is interest?". Some groups, rather than making questions, presented statements such as "The Bank C pays an interest of $ 100 every $ 1000, what would be the interest we could get for $ 10,000?".

Students suggested the expression (Capital × Rate × Time) for compound interest, attempting a "rule of three". The teacher-researcher recommended the study of a simple and compound Capitalization. Students linked the simple capitalization to the Simple Linear Function, and with teacher assistance, they also linked the Compound Capital to the Exponential Function. From a text generated by the teacher, the FE was studied without learning about equations. Then, a problem was presented: how to design a savings plan that generates capital for our children to inherit? All students, even the ones who were parents, agreed to express "*we do not add or withdraw money, as it grows with the interests*".

To the teachers'asking of why they would not add any more money if it could lead to a better investment, they answered that they could not make complex calculations.

They did not add a fee because they had never studied it before, and they did not ponder on how it could be done, because dealing with mathematical questions is not part of the didactic contract. Implicitly, the school for adults seems to consider students have few opportunities to learn, so it aims at facilitating the student access to a graduate certification, minimizing the usefulness of the knowledge obtained. Consequently, the development of the RSP also proved to be impossible in this case.

Some results concerning the different contexts of implementation

The dominant pedagogy and its influence on the attitudes of students that reflect it, made RSP unfeasible in two of the three institutions.

The gestures of the pedagogy of research and questioning the world that students from I1 performed, would have allowed the incipient development of RSP and its dialectics. They studied and researched the question in a quite reflective way. They had interest in the mathematics of the problem and linked it with their daily life. Students also found more mathematical organizations than the students from the other institutions, possibly because these groups had a two-year previous experience in research pedagogy. Adults failed to make genuine questions, corresponding to the monumental pedagogy which offers a vision of mathematics as a discipline distant from real life and reserved for the few. I2 students did not produce any questions throughout the entire implementation. Although they obtained quick access to big amounts of information through the Internet, they could not take advantage of it, because they expected to find "the answer"straightforwardly, continuously demanding the teacher's approval.

In institutions where there was no previous experience, the RSP was unfeasible. Attitudes and habits established in traditional pedagogy were not easily left behind.

Conclusion

A summary of some of the results of the research on the teaching RSP in Argentinian secondary schools within the framework of the Anthropological Theory of the Didactics has been attempted. These studies took place during the last eight years, based on 17 implementations in which (N = 234) students participated.

In some secondary schools, some gestures of the pedagogy of research and questioning the worlf, from the RSP were locally introduced, in a controlled and experimental way. The data analysis shows that there are restrictions originated within the different levels (humanity, civilization, society, institution, pedagogy, didactics, etc.) affecting the ecology of RSP (Llanos & Otero, 2013a, 2013b; Parra, Otero & Fanaro, 2013a , 2013b; Donvito, Sureda & Otero, 2013; Otero et al., 2013).

The implementation of the RSP in relatively controlled settings, allows us to justify the benefits of teaching based on questions rather than answers, as well as the difficulties of its development in conventional classrooms. Such is the case of the path generated by the geometric multiplication of the curves.

Decisions related to the role of students and teacher in the classroom in this and other paths, experience significant drawbacks from the very beginning, due to the fact that in a teaching based on RSP, the teacher abandons his role as "explainer" deeply rooted in traditional pedagogy. This generates a strong initial rejection from students, given that the teacher is no longer the only one responsible for the mathematic activity in the classroom.

Then, teacher faces the challenge of taking over this new role as director of the study and accepting that the class group will decide the course of action. The dialectics of the questions and responses, the problematization attitude, the dialectics of getting in and out of the subject, presents a challenge to the teacher and the students at all educational levels, but the results show that this can be gradually modified.

The decisions made in the *topogenetic* level affect directly on the ecology of the RSP, giving its close relation with the *mesogenetic* process. Thus, in successive implementations of a path with the same generative questions, a richer environment, and more elaborated students' responses have been found. This fact does not show attributable variations on the intelligence of the last groups in relation to the first, one given that the distribution of students in the courses was at random. In case the question was not changed, there were changes in the decisions that the teacher made to give the students more space.

The finished RSP does not fully allow reaching the *pedagogy of research and questioning the world*, though it can be said that some promising gestures have been introduced with some limitations. Due to the inadequacy of the mono-disciplinary RSP to fully develop the pedagogy of research, an RSP which included mathematics and other disciplines was proposed, as in the case of the study of the break-even point of a single property and savings plans, and others cases not reported in the present paper.

The co-disciplinary RSP propose a generalization of the idea of didactic transposition between communities, disciplines, institutions, etc. They are the natural correlative of the *Pedagogy of Research and Questioning of the World*, that establishes why and what for is research needed and what is its basis. The new pedagogy does not seek "scientific literacy", but neither is opposed to it, nor seeks to train scientists or recruit talents for this activity in the classroom. Its purpose is not even to promote admiration and taste feeling about mathematics, but to offer opportunities to understand and to improve its functionality. The premise exceeds the relatively short period of time that involves attending school, in comparison with life duration.

Not all of the dialectics "live" with the same intensity. The difficulties of managing questions and answers have been already set, as well as the complexity of getting into and out of the subject. This didactic gesture is completely alien to the predominant pedagogy in secondary schools and colleges. In these institutions, the study is sequenced, following the disciplinary logic and not the functionality of knowledge. As regards the search for an answer to a problem, it has been shown that we also teach in the illusion of *exhausting subjects* before proceeding to the next one.

The in and out of the subject dialectics, breaks with the habit of the linearity of the disciplinary logic and with the idea that it is necessary and possible to exhaust the study. This entails the difficulty of deciding how much, when and how a certain subject will be studied, according to what is known as black boxes and clear boxes dialectics. For the sake of the RSP survival, these decisions were more guided by the teacher than by the students at the beginning.

In some occasions, there should be special attention not to prevent the potential possibility of the extension of a path. It is important that there isa detailed analysis and a continuous redefinition of the praxeological model of reference (PMR) in the hands of the teacher and the research team. Examples of these analyses have been presented, summarized in diagrams that integrate questions and organizations. Then, the inherent drawbacks in the construction of the PMR and the support that this task provides when making decisions, have shown the need of task work teams, which are rarely available in institutions, basically due to the institutional characteristics of the performance of the teaching profession.

The reading and writing dialectics, has also proved a major challenge. It is very important that students document, analyse and decide what to transcribe and what to rewrite for the preparation of a response. This has been facilitated by the methodology of own data collection, but outside the context of research, it is not a frequent practice. Students are not used to aquestions-based study, and therefore they read and write only what the teacher suggests or dictates. When reading, students have the habit of highlighting certain aspects of the text that they consider relevant, rarely rewriting it as they usually tend to transcribe it. In this instance, it has been useful the written syntheses that the working groups have had to produce apart from their daily records, have been positively useful. These syntheses prove an excellent assessment tool and are also necessary for the dissemination of the answer.

The environment – media dialectic, expresses the set of gestures that allow building the environment, as well as the shared students/teacher responsibility for it. Its originality lies in the fact that students contribute to the environment, not only with their feedback, but also by providing any media they want. This media will be properly analysed as to determine its incorporation or not. This presents an obvious difference as regards traditional pedagogy, where the environment is only the teacher's responsibility, who is at the same time the only media, also in relation within other theories. Students today have a source of immeasurable information access, like the Internet, and this dialectics makes viable the didactic analysis and its use in the class. It has also been shown that the traditional use of technology does not imply any advantage if this is not at the service of a research and questioning pedagogy.

In these eight years of study and research in the framework of the ATD, a documentation of a questions-based teaching as a possible and positive experience has been achieved; and that in such cases, the construction of responses from students has been categorical.

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