Constructing mathematical proofs through abductive reasoning

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Abstract

This study aims to investigate the utilization of abductive reasoning in the construction of mathematical proofs among students. The research adopts a qualitative methodology, involving eight prospective teacher candidates with advanced mathematical proficiency. Data were analyzed by Toulmin's Argumentation Theory. The findings revealed that two participants employed abductive reasoning. These participants applied overcode abduction yet adhered to all the stages of abductive reasoning. Abductive reasoning can significantly contribute to the construction of mathematical proofs and facilitate the development of students' mathematical competencies. The implications of this study include the potential for developing instruments to measure abductive reasoning skills.

Keywords

Constructing proof, mathematical proficiency, abductive reasoning algebraic proofs

Résumé

Cette étude vise à examiner l'utilisation du raisonnement abductif dans la construction de preuves mathématiques chez les étudiants. La recherche adopte une méthodologie qualitative, impliquant huit futurs candidats à l'enseignement ayant des compétences avancées en mathématiques. Les données ont été analysées à l'aide de la théorie de l'argumentation de Toulmin. Les résultats ont révélé que deux participants utilisaient le raisonnement abductif. Ces participants ont appliqué une abduction surcode tout en respectant toutes les étapes du raisonnement abductif. Le raisonnement abductif peut contribuer de manière significative à *la construction de preuves mathématiques et faciliter le développement des compétences mathématiques des élèves. Les implications de cette étude incluent la possibilité de développer des instruments pour mesurer les compétences en matière de raisonnement abductif.*

Mots-clés
Construction de la preuve, compétence mathématique, raisonnement abductif, preuves algébriques

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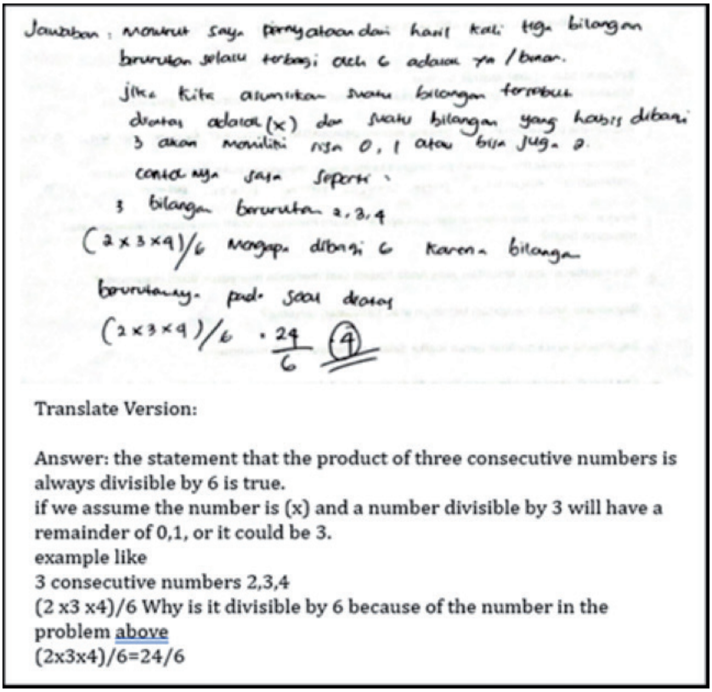
Introduction

Students in mathematics education are expected to attain proficiency in mathematical proof skills, as this is not only a formal requisite in the discipline of mathematics but also an essential foundation for comprehending, developing, and applying mathematical knowledge in diverse contexts (Stylianides et al., 2016). Mathematical proof provides more than merely knowledge of the result of a mathematical theorem or statement; it fosters a profound understanding of the conceptual underpinnings of mathematical structures. By engaging in logic and reasoning, students are invited to validate the truth of a statement based on known premises, resulting in a deeper comprehension of intricate mathematical frameworks.

We have observed that most students tend to employ inductive steps when constructing proofs. However, reported that while students do utilize deductive steps, they often encounter difficulties with the appropriate application of definitions (Siswono et al., 2020). In our study, we presented a proof problem and found that students constructed the proof using inductive steps (Figure 1). The problem posed was: "If a sequence of positive integers with a difference of 1 (e.g., 1, 2, 3) is multiplied, then the result is divisible by 6".

From the analysis of Figure 1, it is evident that students adopt a less formal approach to proof by providing exemplars of numbers. This deficiency indicates that students have not yet mastered the ability to perform formal mathematical proofs. One potential cause is their lack of comprehension of the necessary steps to rigorously prove a mathematical statement. The approach employed by these students is not acceptable in the context of advanced mathematics education, where formal proofs are expected. The errors made by these students demonstrate that they have not fully grasped the fundamental concepts of proof (Miyazaki, 2000). In the author's observations, some students in mathematics education programs struggle with mathematical proof. This difficulty may be attributed to challenges in drawing conclusions from the facts presented in the proof process.

FIGURE 1


Jawaban : Menurut saya pernyataan dari hasil kali tiga bilangan berurutan selalu terbagi oleh 6 adalah ya / benar.
jika kita asumsikan suatu bilangan tersebut diatas adalah (x) dan suatu bilangan yang habis dibagi 3 akan memiliki sisa 0, 1 atau bisa juga 3.
contohnya saja seperti :
3 bilangan berurutan 2,3,4
(2×3×4)/6 mengapa dibagi 6 karena bilangan berurutan pada soal diatas
(2×3×4)/6 = 24/6 4

Translate Version:

Answer: the statement that the product of three consecutive numbers is always divisible by 6 is true.
if we assume the number is (x) and a number divisible by 3 will have a remainder of 0,1, or it could be 3.
example like
3 consecutive numbers 2,3,4
(2 x3 x4)/6 Why is it divisible by 6 because of the number in the problem above
(2x3x4)/6=24/6

Student work results in compiling evidence

The errors made by these students indicate that they have not fully comprehended the foundational concepts of the mathematical proofs they are attempting. The author's observations reveal that some students in mathematics education programs have difficulty performing mathematical proofs. One reason for this difficulty is their struggle to draw conclusions from the given facts in the proof. Table I outlines the proof construction process that can be utilized by students (Weber et al., 2014), providing an overview of the steps necessary for conducting formal mathematical proofs.

TABLE 1

The constructing proofs process

Process	Aspects
Procedural Proofs	Try to build proof by applying the procedure
	Define a specific set of steps
Synthetic Proofs	Manipulating definitions
Semantic Proofs	Trying to understand why a question is correct by checking the representation

Mathematical proofs require valid and precise deductive reasoning. This principle asserts that the reliability of mathematical proofs depends on deductive reasoning, which ensures the correctness of each logical step and the correspondence between the given premises and the resulting conclusion. Thus, deductive reasoning is crucial for ensuring the reliability and certainty of mathematical knowledge and conclusions. Deductive reasoning is a method for deriving logical conclusions based on premises that are accepted or assumed to be true. These premises can be assumptions, axioms, or foundations from which conclusions are drawn (Carreira et al., 2020). When the given premises are true, the conclusions generated through deductive reasoning are necessarily valid. Deductive reasoning involves drawing logical conclusions based on established premises. Premises in this context refer to assumptions, thoughts, and foundations that are considered true. In deductive reasoning, the conclusion will be certain or valid if the given premises are true. Therefore, deductive reasoning is a process that starts from a general statement to a specific conclusion, where the resulting conclusion must be valid. In addition to deductive reasoning, inductive and abductive reasoning can also be employed in the process of mathematical proof.

Inductive reasoning derives specific conclusions from general premises, such as observations, data, or facts. Forms of inductive reasoning include generalizations, analogies, and causal relationships. Through inductive reasoning and the observation of phenomena, one can comprehend the various possibilities that exist (Widadah et al., 2022). Activities that fall under the category of inductive reasoning include: l) concluding one specific case or trait that is then applied to other specific cases; 2) making inferences based on similarities in data or processes; 3) making general inferences based on observations of several data points; 4) using patterns of relationships to analyze a situation.

In the process of mathematical proof, an essential type of reasoning is abductive reasoning. This form of reasoning is employed to discover novel ideas and innovative solutions. Abductive reasoning is a creative and exploratory process that enables individuals to generate new ideas and innovative solutions by proposing plausible explanations for observed phenomena. In other words, abductive reasoning encourages imaginative thinking and can lead to breakthroughs in problem-solving and decision-making contexts (Durand-Guerrier et al., 2012). Through the use of abductive reasoning, individuals can correctly draw connections between first principles in logic and mathematics, which are not necessarily universal generalizations of arithmetic truths or concrete branches of mathematics. Thus, abductive reasoning provides a platform for the exploration of new ideas and innovative approaches to understanding and solving mathematical problems.

In this study, we consider abduction as a specific pattern to obtain the best explanation. According to Peirce (1878, 1903), reasoning patterns can be categorized into three types: deduction, induction, and abduction (Peirce, 1903). In Peirce's framework, ampliative arguments are used to explain arguments whose conclusions refer to existing premises (Bellucci & Bellucci, 2019). This approach strengthens our beliefs in making inferences. Both induction and abduction are ampliative and uncertain, meaning that although the truth of the premises can be accepted unconditionally, it does not guarantee certainty in drawing conclusions. Therefore, conclusions derived from these reasoning patterns need to be further tested and verified.

Abductive reasoning has been adopted as one of the methods to enhance discovery and develop students' creative reasoning (Olsen & Gjerding, 2019; Magnani, 2022). In this context, abduction plays a crucial role as a conduit that connects facts to ideas and theories. Conversely, inductive reasoning is used to explore the relationship between concepts and theories to facts, while deductive reasoning aims to validate a theory. Research by Jeannotte and Kieran (2017) demonstrates that students are often given limited opportunities to develop math-based creative reasoning skills. Therefore, it is imperative to examine how instructional sections in mathematics learning should be structured and utilized to assist in developing students' creative reasoning skills. The deductive approach involves establishing initial premises or assumptions that are then used to draw definite conclusions based on those premises. In contrast, the inductive approach involves observing patterns or case examples, which are then used to infer general rules or principles based on those observations (Rapanta, 2023). Thus, abductive reasoning plays a role in guiding students to formulate hypotheses or initial considerations regarding how they will conduct the proof. This can influence students' choice of using a deductive or inductive approach. The reasoning steps in the proof process can occur circularly, meaning that the reasoning steps can recur at procedural, syntactic, or semantic stages. However, we present each step in this study in one proof process. Additionally, abductive reasoning can assist students in planning the steps or methods to be used in the proof. We present the steps of deductive, inductive, and abductive reasoning in the process of constructing a proof in Table 2.

Table 2

Deductive, inductive, and abductive reasoning steps in the process of developing evidence

The process of constructing proof	Reasoning Steps		
	Deductive	Inductive	Abductive
Procedural Proof	Carry out calculations based on certain rules or formulas	a. Inferring from one special case or trait that applies to another special case. b. Inferring based on data similarity or process	a. Problem identification b. Look for patterns and relationships
Synthetics proof	Construct a direct proof an indirect proof or a proof by mathematical induction	Draw general conclusions based on a set of observed data	Creating a hypothesis
Semantic proof	Draw logical conclusions based on rules of inference, check the validity of arguments, prove, and construct valid arguments	Use patterns of relationships to analyze situations	a. Validation Test b. Communication and justification

According to (Schurz, 2021), induction and abduction are distinct forms of ampliative reasoning that are irreducible to each other. Induction can be considered an umbrella term that covers all types of non-deductive inferences. Both induction and abduction aim to extend our knowledge beyond the limits of available observations. Induction is used to infer something according to a predetermined pattern or plan. However, induction cannot introduce new concepts or conceptual models; it is only capable of transferring existing information. In contrast, abduction is used to infer something that requires an explanation of why it occurred. Abduction has a more proactive role in promoting creative new concepts or models by providing space for the exploration of ideas that have not been previously considered.

By applying abductive reasoning in the context of mathematical proof, students can improve their abstract reasoning ability, find innovative and flexible solutions, and enhance their skills in mathematical modelling and problem-solving (Bellucci & Bellucci, 2019). Abduction involves a generative process that stimulates and nurtures emergent ideas, encourages creativity, and produces novel results in various domains. It is a cognitive process that fosters innovative thinking and can be applied in a variety of fields to generate new ideas, solutions, and theoretical insights. In the context of mathematical proof, the abduction process can involve steps such as problem identification, problem analysis, pattern search, hypothesis formation, and justification.

In this study, abduction is analyzed as part of the argumentation employed by students to substantiate conjectures and compile evidence within the context of algebraic proof. Algebraic proof, involving the manipulation of mathematical expressions such as expansion and factorization to validate statements concerning integers or algebraic terms, is a critical area of study for mathematics education students (Bair & Rich, 2011). Algebraic proof constitutes a foundational aspect of mathematics, as it facilitates the establishment of mathematical truths with certainty (Bair & Rich, 2011). Furthermore, algebraic proof provides a robust basis for the progressive development of mathematics, enabling a deeper exploration and comprehension of underlying mathematical structures and patterns.

Arguments within the framework of abduction are pivotal in the proof construction process when there is a clear connection between the propositions advanced and the compiled proof (Pedemonte & Reid, 2011). According to Toulmin's model, an argument comprises several elements. Firstly, claims are statements or hypotheses proposed in an argument. Secondly, data is employed to furnish support or evidence for the claim. The warrant, an essential component, justifies the use of data to underpin the relationship between data and claims. This warrant can take the form of principles or rules that create a logical bridge between the provided data and the proposed claim. While this foundational structure constitutes the primary components of an argument, supplementary elements are frequently required to elucidate and fortify the argument. Consequently, the deployment of arguments in the context of abduction not only involves the consideration of the claim, data, and warrant but also incorporates additional elements necessary to explicate and reinforce the argument in greater detail.

The impetus for this research comprises several critical facets. Firstly, there is a scholarly interest in elucidating the role and distinctive characteristics of various types of reasoning in the mathematical proof process, alongside identifying the merits and constraints of each type of reasoning. This scholarly curiosity necessitated an investigation focusing on the distinctions among inductive, deductive, and abductive reasoning within the realm of mathematical proof construction (Jeon & Shin, 2022). Although deductive, inductive, and abductive reasoning are all important in mathematical proof, no study has comprehensively communicated the differences between the three types of reasoning. In particular, no study has addressed the differences in deductive, inductive, and abductive reasoning in students as they construct proofs.

Secondly, there is a scholarly endeavor to comprehend the tangible benefits of applying abductive reasoning in mathematics education, particularly in mitigating the complexities students often encounter when engaging with algebraic material. This research seeks to explore the role of abductive reasoning in assisting students to navigate intricate mathematical situations or statements in algebra. Lastly, there is a scholarly objective to deepen the understanding of the abductive reasoning process within the context of mathematical proof. A more profound comprehension of this process aims to develop effective pedagogical methods or strategies to leverage abductive reasoning in mathematical proof, thereby enhancing the efficacy of mathematics education and aiding students in the proficient construction and comprehension of mathematical proofs.

Based on the aforementioned discourse, the research questions in this study are oriented towards three primary aspects. Firstly, the objective is to identify and elucidate the fundamental distinctions between inductive, deductive, and abductive reasoning in the construction of mathematical proofs. Secondly, the research seeks to investigate how abductive reasoning can serve as an efficacious tool for students in addressing complex mathematical situations or statements, particularly within the domain of algebra. Thirdly, the focus is on understanding how the abductive reasoning process can be practically implemented in the construction of mathematical proofs.

This research holds substantial relevance and significance within the field of mathematics education. An in-depth understanding of the role of abductive reasoning in addressing complex mathematical situations or statements in algebra can significantly contribute to the development of more effective and comprehensive instructional strategies. Furthermore, a deeper comprehension of abductive reasoning can aid students in constructing mathematical proofs more proficiently, especially in more abstract contexts. Hence, this research can make a meaningful contribution to the pedagogy and advancement of mathematics education as a discipline.

Methodology

This research employs the Toulmin Argumentation Theory method, as its analytical framework facilitates a detailed explanation of the relationship between argumentation and mathematical proof. The study was conducted at STKIP PGRI Sidoarjo and involved eight sixth-semester mathematics teacher candidates who had completed coursework in number theory. The subjects were selected based on their high mathematical ability, for several reasons: firstly, high mathematical ability encompasses a profound understanding of the fundamental structures and properties of mathematics, which are essential for constructing proofs. Secondly, it involves the capacity to perform rigorous logical analysis of mathematical situations or statements and critically evaluate each step in the proof process. Thirdly, high mathematical ability is necessary for devising innovative and original approaches to formulating proofs. Fourthly, it includes the ability to think abstractly and generalize mathematical concepts effectively. The selection of subjects was based on their Grade Point Average in the fifth semester, which served as an indicator of their mathematical prowess. The criteria for high mathematical ability in this study are if students have a cumulative grade point average above 3.75 on a 4.00 scale. We designed an assignment to be carried out which was discussed with the lecturer of the number theory course. The proof problem posed in this study is illustrated in Figure 2.

FIGURE 2

If a and b are odd number, show that $8|(a^2 - b^2)$

Proving problem

During the proof process, subjects were encouraged to engage in discussions to articulate the arguments supporting their reasoning steps. Upon completing the proofs, subjects were again prompted to elaborate on the proof processes they had undertaken. The collected data was subsequently analyzed using a procedure similar to that of Knipping and Reid (2015). The data analysis consisted of two primary stages. In the first stage, the researcher analyzed the subjects' work using Table 2 as a reference. This analysis focused on identifying the steps of deductive, inductive, and abductive reasoning employed in the proof construction process. In the second stage, the proof process was analyzed using Toulmin's Argumentation Theory. This stage involved examining the reasoning steps used by the subjects in their mathematical proofs and considering the argumentation structure that underpinned the proofs. Thus, the data analysis was comprehensive, taking into account both the reasoning steps and the argumentation framework employed by the subjects in constructing mathematical proofs.

FINDING

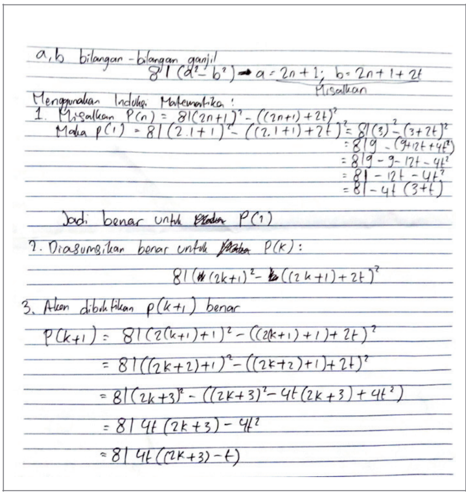
In this section, we describe the research data based on the problem If a and b are odd numbers, show that $8|(a^2 - b^2)$ In is study, we analyzed based on the proof construction process and reasoning steps. In the first part, we present data on the differences between inductive, deductive, and abductive reasoning in mathematical proofs. Here (Table 3) we present the reasoning results of 8 prospective teacher students with high mathematics ability. Subjects F and L used deductive reasoning to construct the proof. The work of subjects F and L can be seen at Figure 3.

TABLE 3

Reasoning of research subjects

Research Subject	Reasoning		
	Deductive	Inductive	Abductive
F	✓		
T		✓	
I			✓
L	✓		
M		✓	
Y		✓	
N		✓	
S			✓

FIGURE 3



Translated Version

$8|(a^2 - b^2) \to a = 2n + 1; b = 2n + 1 + 2t$

Using mathematical induction:

1. Suppose $p(n) = 8|(2n + 1)^2 - ((2n + 1) + 2t)^2$
   Then $p(1) = 8|(2.1 + 1)^2 - ((2.1 + 1) + 2t)^2 = 8|(3)^2 - (3 + 2t)^2$
   $= 8|9 - (9 + 12t + 4t^2)$
   $= 8|9 - 9 - 12t - 4t^2$
   $= 8|-12t - 4t^2$
   $= 8| - 4t(3 + t)$

   So it is true for $p(1)$
2. Assume true for $p(k)$:
   $8|(2k + 1)^2 - ((2k + 1) + 2t)^2$
3. It will be prove $p(k + 1)$ is true
   $p(k + 1) = 8|(2(k + 1) + 1))^2 - ((2(k + 1) + 1 + 2t)^2$
   $= 8|(2k + 2) + 1))^2 - ((2k + 2) + 1) + 2t)^2$
   $= 8|(2k + 3))^2 - ((2k + 3)^2 - 4t(2k + 3) + 4t^2)$
   $= 8|(2k + 3))^2 - ((2k + 3)^2 - 4t(2k + 3) + 4t^2)$
   $= 8|4t(2k + 3) - 4t^2$
   $= 8|4t((2k + 3) - t)$

Subject F's work in constructing evidence Part I

We do not present the work of subject L because their work is the same or identical. Both subjects used deductive reasoning to construct the proof. Figure 3 shows that subject F carried out calculations based on certain rules or formulas, namely by writing "using mathematical induction" which then wrote down the steps. The first step was to suppose $p(n) = 8|(2n+1)^2 - ((2n+1)+2t)^2$ until subject F obtained the conclusion "so it is true for p(l)". The second step is assuming true for $p(k)$: $8|(2k+1)^2 - ((2k+1)+2t)^2$. The third step is to prove that $p(k+1)$ right up to the equation: $p(k+1) = 8|4t((2k+3)-t)$. We relate this to the construction process in procedural proof. Furthermore, subject F drew a logical conclusion based on the inference rule by writing: case l: suppose t is even, $t = 2n$ then writing: case 2: suppose t is odd, $t = 2n+1$ that from cases l and 2 subject F obtained the conclusion: So $8|(a^2 - b^2)$. This can be seen in the following Figure 4.

FIGURE 4

Kasus 1 : Misalkan t genap, $t = 2n$

$$8|4t((2k+3)-t) = 8|8n((2k+3)-2n)$$
$$= 8|8n(2k+3) - 16n$$

Jadi $8|(a^2-b^2)$

Kasus 2 : Misalkan t ganjil, $t = 2n+1$

$$8|4t((2k+3)-t) = 8|4(2n+1)((2k+3)-2(2n+1))$$
$$= 8|4(2n+1)(2k+3) - 2(2n+1)^2$$
$$= 8|4(2n+1)[(2k+3) - \tfrac{1}{2}(2n+1)]$$

Jadi $8|(a^2-b^2)$

Translated Version

Case 1: Suppose t t is even, $t = 2n$

$$8|4t((2k+3)-t) = 8|8n((2k+3)-2n)$$
$$= 8n(2k+3) - 16n$$

So, $8|(a^2-b^2)$

Case 2 : Suppose t is odd, $t = 2n+1$

$$8|4t((2k+3)-t) = 8|4(2n+1)((2k+3)-(2(2n+1))$$
$$= 8|4(2n+1)(2k+3) - 2(2n+1)^2$$
$$= 8|4(2n+1)[(2k+3) - \frac{1}{2}(2n+1)]$$

So, $8|(a^2-b^2)$

Subject F's work in constructing evidence Part 2

Figure 4 shows that subject F draws conclusions based on logical inference rules in the Semantic proof construction process. Thus we can say that subject F constructs proof by mathematical induction in the semantic proof process. We (P) deepened the data by conducting an interview with subject F.

P : How did you construct the evidence?

F : I used mathematical induction with three steps

P : Are you able to come to a conclusion after only three steps?

F : Of course not, I continued with two cases where the variables were replaced with odd numbers and even numbers:

Case l: Suppose t is even, $t = 2n$

$$8|4t((2k+3)-t) = 8|8n((2k+3)-2n)$$
$$= 8n(2k+3) - 16n$$

So, $8|(a^2-b^2)$

Case 2 : Suppose t is odd, $t = 2n + 1$

$$
\begin{aligned}
8|4t\big((2k+3)-t\big) &= 8|4(2n+1)((2k+3)-(2(2n+1)) \\
&= 8|4(2n+1)(2k+3)-2(2n+1)^2 \\
&= 8|4(2n+1)[(2k+3)-\frac{1}{2}(2n+1)]
\end{aligned}
$$

So, $8|(a^2-b^2)$

P : Have you made sure that your arguments are valid and logical?

F : Of course, I re-read the results that I have obtained, and I have confirmed that the mathematical induction that I did was correct.

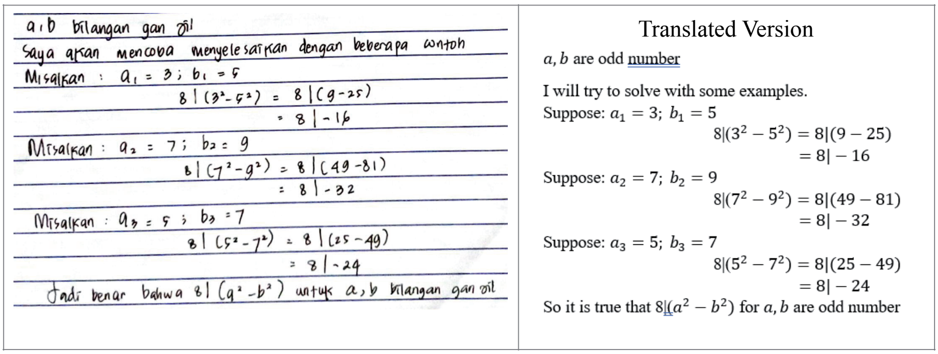
P : Is there a specific strategy to ensure this?

F : No, I do mathematical induction according to routine rules

Based on the results of the interview, it can be seen how the research subject constructs evidence through procedural, synthetic, and semantic evidence processes using inductive reasoning steps or activities as in Table 2.

Four out of eight research subjects used inductive reasoning to construct evidence. Here (Figure 5) we present the description of the inductive reasoning of subjects T, M, N, and Y.

FIGURE 5


a, b bilangan ganjil
Saya akan mencoba menyelesaikan dengan beberapa contoh
Misalkan : $a_1 = 3$; $b_1 = 5$
$8|(3^2-5^2) = 8|(9-25)$
$= 8|-16$
Misalkan : $a_2 = 7$; $b_2 = 9$
$8|(7^2-9^2) = 8|(49-81)$
$= 8|-32$
Misalkan : $a_3 = 5$; $b_3 = 7$
$8|(5^2-7^2) = 8|(25-49)$
$= 8|-24$
Jadi benar bahwa $8|(a^2-b^2)$ untuk a, b bilangan ganjil

Translated Version

a, b are odd number

I will try to solve with some examples.

Suppose: $a_1 = 3$; $b_1 = 5$

$8|(3^2-5^2) = 8|(9-25)$
$= 8|-16$

Suppose: $a_2 = 7$; $b_2 = 9$

$8|(7^2-9^2) = 8|(49-81)$
$= 8|-32$

Suppose: $a_3 = 5$; $b_3 = 7$

$8|(5^2-7^2) = 8|(25-49)$
$= 8|-24$

So it is true that $8|(a^2-b^2)$ for a, b are odd number

Result of subject T's work in constructing evidence

We only present the work of subject T because the work of the other three is identical. All four subjects used inductive reasoning to construct the proof. Figure 5 shows that subject T carried out drawing conclusions from one special case or trait applied to another special case, namely by making an initialization. $a_1, a_2, a_3, b_1, b_2, b_3$ and continued by making conclusions. Subject T made inferences based on the similarity of data or processes, namely constructing evidence by trying several examples of numbers (3, 5, 7, 9) which are substituted into $8|(a^2 - b^2)$. Used on these two things, we say that subject T performed a procedural process on the construction of evidence. Subject T made a general conclusion based on several observed data, namely by providing three examples of cases then subject T made observations of the three examples written down to conclude that it is true that $8|(a^2 - b^2)$ for a, b are odd numbers. In this case, we say that subject T performed a synthetic proof process in the proof construction process. Subject T also used relationship patterns to analyze the situation, namely by writing "a, b are odd numbers" which we further said that subject T performed semantic proof in the proof construction process. To deepen the data we obtained, we (P) interviewed with subject T.

P	:	How did you construct the evidence?
T	:	I constructed the proof by giving examples
P	:	How would you apply the example you gave to other cases?
T	:	Taking another example
P	:	How do you use similarities to make inferences?
T	:	I substituted it into $8|(a^2 - b^2)$ so I can conclude that $8|(a^2 - b^2)$ is true for a, b are odd number
P	:	How can you infer something general from the data?
T	:	Through observing the examples I proposed, I was able to make conclusions.
P	:	How do you ensure that the generalizations you make are supported by data?
T	:	I identify patterns according to the data or examples I choose

Based on the results of the interview, it can be seen how the research subject constructs evidence through procedural, synthetic, and semantic evidence processes using inductive reasoning steps or activities as in Table I.

Next, we present the abductive reasoning of two subjects out of 8 subjects, namely subjects S and G. In this second part, we describe based on Toulmin's representation with discussion referring to the proof construction process and abductive reasoning steps (Table 4).

Based on the argument, it can be seen that subject S used the linearity property to construct the proof. Subject S analyzed the problem by writing a and b are odd numbers, claim: odd number: (2k+1) or (2k+1)+2t, *warrant*: If a|b and a|c then a|(b+c) and a|bc. This warrant is a hypothesis. Subject S validated, communicated, and justified by saying "I think it works when I replace variables with numbers, but I don't write it down". Thus, it can be said that subject S's proof construction process is in accordance with the steps of abductive reasoning (Table 5).

TABLE 4

Subject S's abductive reasoning based on Toulmin's representation

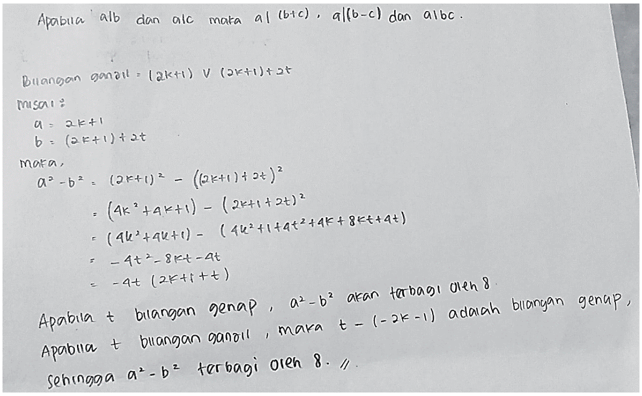
Argumentation	Toulmin's representation
Subject S did not know with his own reasoning S: I'm not sure about k and t, do they have to be whole numbers?	**D:** *a and b are odd number* → **C:** *odd number: (2k+1) and (2k+1) + 2t* **W:** *linearity properties (If* $a\|b$ *and* $a\|c$ *then* $a\|(b+c), a\|(b-c)$ *and* $a\|bc$)
Q: If you had doubts, how did you proceed with the proof? Also, you wrote "or", don't you use both? *Subject S realized that both numbers would be used, but should have used "and"* S: oh yes, I should have used the word "and". I continued according to the instructions in the problem, namely substitution (2k+l) and (2k+l) +2t ke $a^2 - b^2$ without considering whether k and t must be integers P; What did you get? S: I think it is proven that If a and b are odd numbers, then it must be $8\|(a^2 - b^2)$" P: How can you say it is proven when there is no explanation in your writing *Subject S looked back at the results he obtained* S: I think it works when I replace the variable with a number, but I don't write it down	*Subject S generalized odd numbers with (2k+l) and (2k+l)+2t without thinking whether k and t should be integers. Subject S also wrote" or", which be have been "and"*  Apabila a|b dan a|c maka a|(b+c), a|(b-c) dan a|bc. Bilangan ganjil = (2k+1) V (2k+1)+2t misal: a = 2k+1 b = (2k+1)+2t maka, $a^2-b^2 = (2k+1)^2 - ((2k+1)+2t)^2$ $= (4k^2+4k+1) - (2k+1+2t)^2$ $= (4k^2+4k+1) - (4k^2+1+4t^2+4k+8kt+4t)$ $= -4t^2-8kt-4t$ $= -4t(2k+1+t)$ Apabila t bilangan genap, a^2-b^2 akan terbagi oleh 8. Apabila t bilangan ganjil, maka t-(-2k-1) adalah bilangan genap, sehingga a^2-b^2 terbagi oleh 8. // Translated version If a|b and a|c then a|(a+b), a|(b-c) and a|bc Suppose: $a = 2k + 1$ $b = (2k + 1) + 2t$ Then, $a^2 - b^2$ $= (2k + 1)^2 - ((2k + 1) + 2t)^2$ $= (4k^2 + 4k + 1) - (2k + 1 + 2t)^2$ $= (4k^2 + 4k + 1) - (4k^2 + 1 + 4t^2 + 4k + 8kt + 4t)$ $= -4t^2 - 8kt - 4t$ $= -4t(2k + 1 + t)$ If t is an even number, $(a^2 - b^2)$ will be divided by 8 If t is an odd number, then $t - (-2k - 1)$ is an even number so that $(a^2 - b^2)$ is divisible by 8

TABLE 5

Subject G's abductive reasoning based on Toulmin's representation

<table>
<tr><th>Argumentation</th><th>Toulmin's representation</th></tr>
<tr><td>

Subject G identified the problem by writing: suppose a= 2n+l and b=(2n+l)+2q, by previously writing that the number is odd:2n-l or 2n+l
S: I used 2n+l to continue the proof.
P: Do you believe the steps you took will help you construct the evidence?

S: Yes, because understanding the root of the problem will help me identify a solution.

Subject S made a hypothesis, namely "If a|b and a|c then a|(mb+nc) for every integer m and n

P: How do you hypothesize?
S: Keeping the linearity property in mind, to ease the proof construction

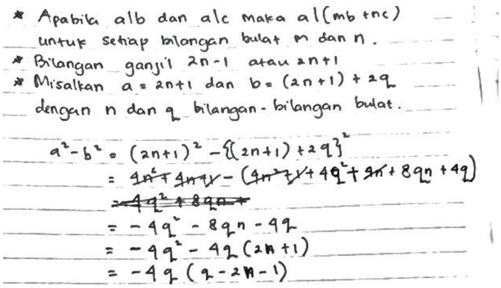
P; What did you get?
S: I think it is proven that If a and b are odd numbers, then it follows that
$8|(a^2-b^2)$"

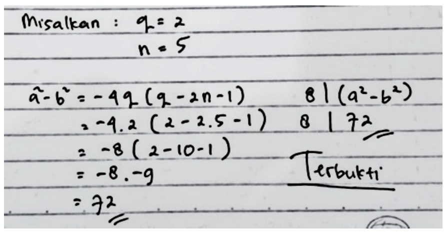
P: How can you be sure?
S: Yes, I'm sure because I replaced the variables with numbers, but I didn't write them down.

Subject S validated the proof by initializing q and n, i.e. q=2, n=5.

</td><td>

D: *a and b are odd number* → **C**: *odd number: (2n+1) and (2n+1) + 2q*
W: *linearity properties (If a|b and a|c then a|(mb+nc) for every integer m and n)*

Subject S generalized odd numbers with (2n+l) and (2n+l)+2q dengan menuliskan bahwa n and q should be integers. Subject S also wrote" and", karena keduanya digunakan

* Apabila a|b dan a|c maka a|(mb+nc)
untuk setiap bilangan bulat m dan n.
* Bilangan ganjil 2n-1 atau 2n+1
* Misalkan a = 2n+1 dan b = (2n+1) + 2q
dengan n dan q bilangan-bilangan bulat.
$a^2-b^2 = (2n+1)^2 - \{(2n+1)+2q\}^2$
$= \cancel{4n^2+4n+1} - (\cancel{4n^2+1}+4q^2+\cancel{4n}+8qn+4q)$
$\cancel{-4q^2+8qn+}$
$= -4q^2 - 8qn - 4q$
$= -4q^2 - 4q(2n+1)$
$= -4q(q-2n-1)$

Translated version
If a|b and a|c then a|(mb+nc) for every integer m and n
Odd number: 2n-l or 2n+l
Suppose a=2n+l and b=(2n+l)+2q, with n and q are integers
$a^2-b^2=(2n+1)^2-[(2n+1)+2q]^2$
$=(4n^2+4n+1)-(4n^2+1+4q^2+4n+8qn+4q)$
$=-4q^2-8qn-4q$
$=-4q^2-4q2(n-1)$
$=-4q(q-2n-1)$
Subject S continued the proof construction by memorizing q and n

Misalkan : q = 2
n = 5
$a^2-b^2 = -4q(q-2n-1)$ $8|(a^2-b^2)$
$= -4.2(2-2.5-1)$ $8|72$
$= -8(2-10-1)$ Terbukti
$= -8.-9$
$= 72$

Translated Version:
Suppose: $q=2, n=5$
$a^2-b^2=-4q(q-2n-1)$
$=-4.2(2-2.5-1)$
$=-8(2-10-1)$
$=-8.-9$
$=72$
$8|(a^2-b^2)=8|72$ (Proven)

</td></tr>
</table>

Based on the argument, it can be seen that subject G used the linearity property to construct the proof. Subject G analyzed the problem by writing s*uppose a=2n+1 and b=(2n+1)+2q, with n and q are integers* claim: odd number: (2k+1) atau (2k+1)+2t, *warrant*: *If a|b and a|c then a|(mb+nc) for every integer m and n*. *Warrant* this is a hypothesis. Subject S validates, communicates, and justifies writing it down". Thus it can be said that subject S's proof construction process is in accordance with the steps of abductive reasoning.

DISCUSSION

Based on the results of the analysis, three primary aspects are discussed. First, there is a notable difference in reasoning among research subjects with high mathematical ability|. Subjects utilizing inductive reasoning in constructing mathematical proofs tend to observe specific data or patterns to infer general rules or principles. Conversely, subjects employing deductive reasoning rely on known or generally accepted premises to reach definitive and logical conclusions. Those using abductive reasoning typically formulate hypotheses or possible explanations based on existing observations or data. The fundamental distinction among these types of reasoning lies in their methods of generating mathematical conclusions or proofs: inductive reasoning infers general principles from specific patterns or data, deductive reasoning derives definite conclusions from known premises, and abductive reasoning hypothesizes explanations based on available observations or data.

Second, some subjects in this study constructed proofs using various types of reasoning, including deductive, inductive, and abductive. Abductive reasoning, for instance, is an initial stage in the cognitive process where one formulates hypotheses or conjectures based on available information (Williamson, 2016). This creative phase permits the proposal of diverse solutions or explanations in response to a problem or phenomenon. Following the presentation of hypotheses or conjectures, the subsequent step involves developing deductions or statements derived from those premises or conjectures, employing logic to reach coherent and consistent conclusions. These deductions or conclusions are then generalized based on the inferred premises, leading to broader generalizations or rules (Stylianides et al., 2016). This involves extrapolating existing information to formulate broader generalizations or universal rules.

The process of abductive reasoning can be applied in constructing mathematical proofs through steps such as observation and identification of patterns, formulation of hypotheses, consideration of alternative solutions, exploration of creative solutions, and evaluation of the results obtained (Beirlaen & Aliseda, 2014). The proof construction process begins with the observation and identification of patterns within mathematical situations or statements. Students employ abductive reasoning to discern possible relationships or patterns among mathematical elements. They then formulate hypotheses or assumptions regarding properties or relationships that may hold, involving speculation or prediction based on the identified patterns. In building proofs, students often explore creative solutions to substantiate their hypotheses. The abductive reasoning process involves reflection and evaluation of the constructed evidence (Font et al., 2013). Students assess whether the proof meets mathematical standards and whether the solution aligns with the proof's objectives. Mathematical communication also influences the proof construction process, as students are expected to articulate the steps of their proof arguments clearly, ensuring comprehensibility by others.

Third, abductive reasoning can positively impact students' learning experiences, particularly in the context of algebraic material. Students can employ abductive reasoning to formulate hypotheses or potential solutions. Through the application of abductive reasoning, students can identify and analyze patterns emerging in complex mathematical situations (Cramer-Petersen et al., 2019). This aids in deepening their understanding of the relationships among variables, algebraic structures, and mathematical properties. By formulating hypotheses or explanations based on existing evidence, students can gain a more profound comprehension of complex algebraic concepts. Additionally, designing and testing hypotheses and gathering supporting evidence enhance students' critical thinking skills. Abductive reasoning fosters careful consideration of solutions and clearer articulation of their thought processes (Olsen & Gjerding, 2019).

In this study, both students demonstrated an overcode abduction in Toulmin's representation. They utilized the linearity property as the basis for formulating hypotheses. The representations used by both students are depicted in Figure 6.

FIGURE 6

D: *a and b are odd number* → **C**: *odd number: (2k+1) and (2k+1) + 2t*	**D**: *a and b are odd number* → **C**: *odd number: (2n+1) and (2n+1) + 2q*
W: *linearity properties (If* $a\|b$ *and* $a\|c$ *then* $a\|(b+c), a\|(b-c)$ *and* $a\|bc)$	**W**: *linearity properties (If a|b and a|c then a|(mb+nc) for every integer m and n)*

Subject hypothesis

During the process of overcode abduction, both students employed only one rule. They proceeded with the proof construction process based on logical reasoning. Although one student intended to replace variables with numbers, this step was not documented in writing. This action aimed to evaluate the validity of the obtained evidence.

The implications of this study include the potential for developing instruments to measure abductive reasoning skills and the revision of learning strategies. The findings of this study can serve as a foundation for enhancing instructional approaches. Integrating abductive reasoning into the curriculum can improve students' capacity to construct mathematical proofs, a crucial competency in higher education.

Conclusion

Based on the research results that have been presented, we can conclude that inductive, deductive, and abductive reasoning have differences in the way they construct mathematical proofs. Inductive reasoning involves observing specific patterns or data to draw a general conclusion. Deductive reasoning uses known premises to reach a definite conclusion. Abductive reasoning creates hypotheses or possible explanations based on observations or available data. The process of abductive reasoning in constructing mathematical proofs involves observing complex mathematical patterns or phenomena, forming a hypothesis or plausible explanation to explain the pattern, testing, and strengthening hypotheses through deductive and inductive reasoning, and accepting or rejecting the hypothesis based on consistency with existing mathematical evidence. The abductive reasoning process involves reflection and evaluation by considering the consistency of the evidence with the hypothesis. Abductive reasoning helps students deal with complex mathematical situations or statements in algebraic material because: enables them to discover hidden patterns or rules behind complex problems, provides a framework for formulating hypotheses or plausible explanations based on their understanding of the material, and provides a foundation for understanding and evaluating various mathematical concepts and strategies in an algebraic context. With abductive reasoning, students not only understand concepts formally, but also actively engage in explorative and creative processes. Thus, abductive reasoning can play a role in mathematical proof construction and can help students develop better mathematical skills. This study's results could be applied to other subjects and problems. However, further investigation of abductive reasoning in the context of algebraic proof is needed.

We suggest that researchers interested in studying abductive reasoning identify complex mathematical situations or statements that will be the focus of their research. This step is critical to clarify the mathematical context in which abductive reasoning is applied, enabling a deeper understanding of how students employ this reasoning in addressing complex problems. Additionally, we recommend expanding the research subjects to enrich the data obtained and produce a more comprehensive understanding of the use of abductive reasoning in mathematics education.

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