Comparing static and dynamic representations: Secondary mathematics teacher candidates' specialized content knowledge, argumentation schemes, and technological perspectives

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Abstract

This preliminary study investigates how teacher candidates' (TCs) specialized content knowledge (SCK) is associated with their technological perspectives and argumentation schemes in interpreting dynamic and static geometry representations. An embedded unit single case study design was employed to analyze TCs' engagement with two mathematical tasks. Qualitative content analysis of TCs' responses highlighted patterns in their evaluations of invalid mathematical arguments and insights regarding different technologies. Preliminary findings suggest an interrelation between TCs' SCK and argumentation schemes. The study proposes that dynamic geometry tasks that encourage analytical argumentation can support TCs' content knowledge and argumentation skills.

Keywords

Teacher candidates, specialized content knowledge, technological perspectives, argumentation schemes, dynamic geometry

Résumé

Cette étude préliminaire examine comment les connaissances spécialisées (SCK) des futurs enseignants (TC) sont associées à leurs perspectives technologiques et à leurs schémas d'argumentation dans l'interprétation des représentations géométriques dynamiques et statiques. Un modèle d'étude de cas unique à unité intégrée a été *utilisé pour analyser l'engagement des TC dans deux tâches mathématiques. L'analyse qualitative du contenu des réponses des TC a mis en évidence des schémas dans leurs évaluations des arguments mathématiques non valides et des idées concernant les différentes technologies. Les résultats préliminaires suggèrent une interrelation entre le SCK et les schémas d'argumentation des TC. L'étude propose que les tâches de géométrie dynamique qui encouragent l'argumentation analytique puissent soutenir la connaissance du contenu et les compétences d'argumentation des TC.*

Mots-clés

Futurs enseignants, connaissances spécialisées, perspectives technologiques, schémas d'argumentation, géométrie dynamique

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Introduction

Mathematics education is enriched by teaching strategies and tools that enhance students' reasoning, problem-solving, and collaborative abilities. Research highlights multiple benefits for students, including developing a strong foundation for deductive reasoning, which is essential for advanced mathematical thinking (Jones, 2000). Additionally, engaging students in mathematical modeling activities improves their ability to represent, analyze, and solve real-world problems (Greefrath et al., 2018). Such practices also foster collaboration and creative thinking as students work together to navigate complex problem-solving tasks, an approach that has significantly enhanced learning outcomes (Granberg & Olsson, 2015). These benefits underscore the importance of integrating innovative approaches that support cognitive and social development in mathematics classrooms. Despite advancements in teaching strategies and tools, many teacher preparation programs inadequately equip teacher candidates (TCs) with the skills to integrate technology into mathematics classrooms effectively (Admiraal et al., 2015; Jones, 2001; Kartal, 2022). This gap is particularly concerning given the increasingly prominent role of technology in fostering mathematical reasoning, problem-solving, and collaboration among students. To address this, we designed a sequence of GeoGebra-integrated geometry tasks to engage our TCs preparing to teach secondary school mathematics in making sense of high school students' mathematical thinking in a technology-oriented mathematics classroom.

Dynamic Geometry Software packages (DGS) (e.g., GeoGebra, Geometry Pad, and Geometer's Sketchpad) provide visual representations that mediate students' learning by offering new mental actions in support of content knowledge development (Hollebrands, 2007; Radović et al., 2020). Research indicates that engaging students in explorations of mathematical ideas using DGS supports their mathematical thinking, strengthens their conceptual understanding of advanced mathematical topics, and the quality of their argumentation and proving skills (Birgin & Uzun Yazıcı, 2021; Jones, 2000; Sinclair & Yurita, 2008). McCulloch et al. (2021) advocated that mathematics methods courses in teacher education programs should provide opportunities for mathematics TCs to experience learning mathematics with the support of DGS and engage TCs in thinking about how DGS might support students' mathematical reasoning. Incorporating more of these programs within teacher preparation programs allows TCs to become more prepared to enter or create more technology-rich classrooms.

The fast growth of modern technologies in support of mathematics teaching and learning requires adequate preparation of teachers, which exposes teachers (including TCs) to new mathematical tasks and helps them think about the nature of technology-based classroom mathematical activities that support pedagogical goals. However, despite the massive affordances of modern technologies to provide students with opportunities for visualization, manipulation, and exploration of geometrical figures and mathematical concepts, their usage is not widespread in teacher preparation programs. McCulloch et al. (2021) further recognized that most teacher education programs do not provide opportunities for secondary mathematics TCs to understand students' mathematical reasoning with technologies. Our work responds to this challenge by designing a sequence of GeoGebra-integrated geometry tasks to engage TCs in making sense of high school students' mathematical thinking in a technology-oriented mathematics classroom.

The literature points out that mathematics teachers who develop interconnected content knowledge during teacher education are likely to make effective pedagogical decisions and provide better learning opportunities for students (Ko et al., 2017; Steele, 2003). Visual representations provided by technological tools mediate prospective teacher learning and offer new mental actions for content knowledge construction (Alqahtani & Powell, 2017). Therefore, examining the factors supporting or limiting TCs' specialized content knowledge (SCK) within differently afforded technological environments is essential.

Similarly, mathematics teachers and teacher candidates benefit from instructional practices that strengthen their own mathematical and pedagogical skills. For instance, engaging in activities that require spatial reasoning enhances teachers' spatial visualization abilities, a skill crucial for effective mathematics instruction (Güven & Kosa, 2008). Moreover, exposure to teaching strategies aligned with best practices for grades 6-12 equips teacher candidates to clearly understand how mathematics should be taught and learned, fostering their confidence and competence in the classroom (Hall & Chamblee, 2013). Together, these professional benefits ensure educators are well-prepared to create dynamic and effective learning environments, promoting more profound understanding and engagement among their students.

The interplay between teachers' technological perspectives, the characteristics of preservice teachers' (PSTs') argumentations, and their specialized content knowledge (SCK) within dynamic geometry environments remains an underexplored area of research. While existing studies have examined the potential of dynamic geometry tools to enhance mathematical reasoning and engagement (Moreno-Armella et al., 2008; Ng, 2016; Sinclair & Yurita, 2008), limited attention has been given to how these tools shape PSTs' argumentation styles and their conceptual understanding. Investigating this relationship could provide valuable insights into optimizing instructional strategies and technological integration to support robust mathematical learning and discourse.

Previous research focused on exploring the roles of dynamic and static representations in knowledge construction and teacher discourse (Sinclair & Yurita, 2008). However, the relationship between teachers' technological stances, the characteristics of TCs' argumentations, and their SCK in dynamic geometry environments has not been extensively investigated. To address this gap in the literature, this preliminary study examines the following research question: *How is TCs' SCK associated with their technological perspectives and argumentation schemes in the context of interpreting dynamic and static geometry representations?*

Theoretical framework

Dynamic and static representations play distinct yet complementary roles in knowledge construction and teacher discourse. Research highlights that reasoning with dynamic representations enhances generalization skills by enabling learners to explore, verify, and compare conceptual relationships in a fluid, interactive manner (Fahlgren & Brunström, 2014; Yao & Manouchehri, 2019; Yerushalmy, 1993). These representations encourage deeper engagement with abstract ideas by illustrating changes and patterns over time. Conversely, static representations support the development of predictive reasoning and the formulation of hypotheses or theories. They provide a structured framework for procedural communication and facilitate clarity in conveying complex topics. These representation types enrich instructional practices, fostering versatile reasoning and discourse strategies.

Successful mathematics teachers know how to use multiple representations to support students' mathematical learning and motivation (Kuntze & Dreher, 2014). To achieve this (i.e., to identify and utilize different mathematical representations to facilitate student learning), mathematics teachers must demonstrate interconnected domain-specific mathematical knowledge and appropriate technological perspectives to deal with various representations (Dreher & Kuntze, 2015).

Interconnected mathematical knowledge is critical for fostering students' understanding of mathematical concepts and addressing their learning needs. It equips teachers to identify the mathematical roots of students' struggles, misconceptions, and unconventional problem-solving methods while enabling them to respond to inquiries about commonly used strategies and argumentations. Moreover, this knowledge aids in interpreting representations—static or dynamic—with or without technology, thereby supporting effective instructional practices (Algahtani & Powell, 2017; Depaepe et al., 2015; Hill et al., 2005). Rooted in integrating Pedagogical Content Knowledge and Subject Matter Knowledge, including Common Content Knowledge and Specialized Content Knowledge, integrated mathematical knowledge empowers teachers to make sense of hypothetical errors, construct and justify algebraic relationships, and generalize mathematical principles grounded in observed cases. This comprehensive understanding enhances teachers' ability to make informed pedagogical decisions, improving the overall quality of mathematics instruction (Algahtani & Powell, 2017; Lesseig, 2016; Steele, 2013).

Mathematical Knowledge for Teaching (MKT) refers to the mathematical knowledge required for effective teaching (Ball et al., 2008). The subject matter component of MKT includes both common content knowledge and specialized content knowledge (SCK), which teachers uniquely need to analyze, interpret, and represent mathematical concepts for student understanding (Steele, 2013). SCK enables teachers to unpack mathematical ideas, diagnose student misconceptions, and design instructional strategies that promote deep learning. This type of knowledge ensures that educators not only understand mathematics themselves but can also convey it effectively to diverse learners. Ball and her colleagues (2008) underline the following teaching tasks that should be demonstrated as a result of mathematics teachers' SCK: "responding to students' why questions, [...] recognizing what is involved in using a particular representation, linking representations to underlying ideas and other representations, [...] evaluating the plausibility of students' claims (often quickly), giving or evaluating mathematical explanations, [...] selecting representations for particular purposes" (p. 400). In line with this quote, connecting different representations and supporting and evaluating students' mathematical argumentations are essential components of mathematics teachers' SCK. Teachers can move students beyond procedures toward a more profound and robust understanding by fostering argumentation, which could then contribute to their SCK.

Mathematical argumentation is the process of justifying claims through evidence and explanation of reasoning, fostering critical thinking and more profound understanding (Inglis et al., 2007; Reuter, 2023). Toulmin's model of argumentation (2014) structures mathematical discourse with components such as claims, evidence, and reasoning. According to Toulmin's model, claims assert mathematical truths, the evidence provides examples or data supporting the claims, and reasoning establishes the logical connection between claims and evidence. Sowder and Harel (1995) present three categories to organize proof schemes: externally-based, empirical, and analytical. Externally-based proof schemes are justifications dependent on authority, such as a teacher or a textbook. Empirical proof schemes rely on specific examples of observed patterns, such as proving an argument using drawings or figures. Analytical proof schemes involve justifications that rely on logical reasoning and general principles; students can build arguments within a structured system of definitions and theorems. These methods ensure that students develop robust mathematical reasoning skills by engaging with different forms of justification.

Technology plays an evolving role in mathematics education, shaping how teachers and students interact with mathematical content (Goos et al., 2003). As SCK is the mathematical content knowledge enabling mathematics teachers to identify different representations and link them to mathematical ideas, teachers' mathematics-specific technology knowledge becomes an essential part of their SCK (Zambak & Tyminski, 2019). "Even if [...] teachers gain the knowledge to implement [technology in their classrooms], they still need [the] confidence to implement [technology] within their specific contexts" (Ertmer et al., 2010, p. 277). In this respect, considering teacher knowledge and their beliefs and perspectives is necessary for teacher preparation, professional development, and change.

Technological perspectives of teachers are defined as the pedagogical perspectives "about the role of technology in learning" (Goos & Bennison, 2008, p. 105). Several frameworks have been used to categorize teachers' technology perspectives (Chen, 2011; Goos et al., 2003; Thurm & Barzel, 2022). For example, Thurm and Barzel (2022) listed four modes of instructional technology perspectives: using technology for multiple representations, discovery learning, individual learning, practicing, and reflection. Differently, Goos et al. (2003) classified teacher technology perspectives within four metaphors: as an extension of self, as a partner, as a servant, and as a master. When technology acts as an extension of self, it enhances a teacher's ability to present dynamic mathematical representations. As a partner, technology actively engages students in problem-solving. As a servant, it performs routine tasks such as calculations without necessarily fostering deep conceptual understanding. When technology becomes a master, it can lead to overreliance, where students depend on digital tools rather than developing independent reasoning skills. Our study adapted these four metaphors to categorize TCs' technology perspectives.

Methodology

Our study with 15 TCs enrolled in a secondary mathematics methods course took place in a private liberal arts university in the mid-Atlantic region of the U.S. TCs participated in a two-week study engaging with mathematical tasks designed to deepen their conceptual understanding and instructional strategies. Data for the study consisted of (1) pre-class assignments (i.e., TCs' analyses of hypothetical secondary students' arguments), (2) class observation notes, and (3) TCs' post-class reflections about their perspectives regarding different technological representations. We coded TCs' responses to pre-class assignments and examined whether or not each TC validly evaluated hypothetical secondary students' geometrical reasoning with attention to geometrical details amplified by the task. Additionally, we categorized TCs' post-reflections to differentiate their technological perspectives.

Following the definition of SCK by Ball et al. (2008), two tasks were developed for which TCs evaluated the validity of hypothetical student claims and unpacked the mathematics involved in static and dynamic representations. The tasks required TCs to (a) analyze students' mathematical thinking and identify mathematical errors in student explanations, (b) make sense of the mathematics behind visual models, and (c) appraise their experiences with static and dynamic geometry representations in the context of teaching secondary mathematics with technology. For Task 1 (dynamic representation), TCs interacted with an applet built into GeoGebra (https://www.geogebra.org/m/pHUfRu7V) and examined one secondary school student's response to a geometry problem with the Triangle Inequality Theorem (Figure 1). For Task 2 (static representation), TCs were asked to attend to mathematical facts and principles behind a group of geometrical figures and guided to interpret a secondary student's response to a problem dealing with the law of cosines (Figure 2). For both tasks, TCs were asked to (i) interpret representations in building geometrical generalizations and (ii) reflect on the impact of various technological representations in constructing and comparing generalizations.

FIGURE 1

Experiment with the GeoGebra applet for a few minutes to familiarize yourself with equality in geometry. Now, think about the following scenario: *Marco, a seventh-grade student, says he can draw a triangle with three sides measured at 3, 4, and 9 feet.*

A. What do you think about Marco's statement? Is it possible to draw such a triangle? Why or why not? Please clearly write your explanation.
B. Based on your observations and experimentations with the applet, what should *a* be equal to in terms of *b* and *c* for any given ABC triangle?
C. Do you think the relationship you hypothesized in (B) is mathematically valid for any triangle? Why or why not?

Task 1 for Dynamic Representation

FIGURE 2

Explore the representations provided in the Appendix for a few minutes to make sense of an equation in trigonometry. Now think about the following scenario: *For a triangle ABC, the following was given: A = 20, C = 60, c = 4 feet, b = 6 feet. Find |a|. Catherine, a tenth-grade student, solved this problem as follows:*

$$a^2 = b^2 + c^2 - 2 \times b \times c \times cos\ (A)$$
$$a^2 = 36 + 16 - 2 \times 6 \times 4 \times cos\ (60°)$$
$$a^2 = 52 - 48 \times \frac{1}{2} = 52 - 24 = 28$$

I know that for any triangle, $\underline{a}^2 = \underline{b}^2 + c^2 - 2 \times b \times c \times cos\ (A)$*. Then I plugged the givens. The length of a is* $2\sqrt{7}$*.*

A. What do you think about Catherine's work? Is Catherine's work and conclusion mathematically correct? Why or why not?
B. Is Catherine's last statement mathematically correct? How do you know that? Based on your explorations with the representations, what should *a* be equal to in terms of *b*, *c*, and *angle A* for any given ABC triangle?
C. Do you think the relationship you hypothesized in (B) is mathematically valid for any triangle? Why or why not?

Task 2 for Static Representation

The tasks were aligned with key mathematics content standards, including 6.G.A.1 (Geometry: Solve real-world and mathematical problems involving area, surface area, and volume), HSG.SRT.B.5 (Similarity and Right Triangles: Use similarity criteria to solve problems), HSG.SRT.C.7 (Trigonometry: Apply trigonometric ratios in non-right triangles), and HSF.TF.C.9 (Trigonometric Functions: Apply functions to model periodic phenomena). Additionally, the study emphasized Mathematical Practice standards, MP2 (i.e., Reason abstractly and quantitatively), and MP3 (i.e., Construct viable arguments and critique the reasoning of others), fostering candidates' ability to apply mathematical reasoning and communication in instructional contexts.

The study employed an embedded unit single case study design (Yin, 2003) to analyze teacher candidates' (TCs) engagement and learning across the two mathematical tasks. Data analysis involved qualitative content analysis of TCs' responses, focusing on patterns in their evaluations of invalid mathematical arguments and insights they shared regarding different technologies and representations they engaged with (Mayring, 2015). TCs' responses were scored using rubrics developed for three key constructs. The first construct assessed TCs' Level of Specialized Content Knowledge (SCK), categorizing responses as Robust SCK (2), Limited SCK (1), or Lack of SCK (0) (Steele, 2013). The second construct examined TCs' Argumentation Schemes, identifying the presence (1) or absence (0) of analytical, empirical, or externally based argumentation in their reasoning (Sowder & Harel, 1998). These argumentation schemes highlight the varying levels of depth and rigor in mathematical reasoning and discourse. The third construct evaluated TCs' Technology Perspectives, classifying their approaches as Experiment-oriented (3), Learning-oriented (2), Accuracy-oriented (1), or Teaching-oriented (0) (Goos et al., 2003). This comprehensive scoring framework (Table 1) provided insights into TCs' mathematical understanding, use of technology, and argumentation skills.

TABLE 1

Scoring TCs' SCK, argumentation schemes, and technological perspectives

Labels	Characteristics
SCK	
Robust SCK (2)	The response is accurate and almost always includes mathematically valid and relevant intermediate and final claims. Justification mainly uses general mathematical principles to make a general argument.
Limited SCK (1)	The response is partially accurate. Some of the intermediate claims are mathematically valid and relevant but require additional support for clarity. Justification mainly uses a form of partially supported empirical argument. OR, the response is partially accurate. Some of the intermediate claims are mathematically valid and relevant but require additional support for clarity. Justification is mainly based on partially supported observations.
Lack of SCK (0)	The response is partially accurate. Some intermediate claims are mathematically valid and relevant, but no justification is provided. OR the response is inaccurate, missing, or vague. Some intermediate or final claims are erroneous, mathematically invalid, or irrelevant.
Argumentation Schemes	
Analytical Argumentation (A)	The generalized response demonstrates a deep understanding of algebraic rules and manipulations. Its justifications are logical and well-structured and provide a strong foundation for mathematical proofs.
Empirical Argumentation (E)	The response relies on one or more numerical cases to support conclusions. While specific examples are used effectively, the argument lacks generalization and broader applicability.
Externally-based Argumentation (EB)	The response includes a superficial reference to a mathematical term or theory without providing any justification. The argument lacks depth, analytical reasoning, and logical coherence.
Technological Perspectives	
Experiment-oriented	The response reflects the participant's view that technology is an extension of their mind, often emphasizing its ability to support experimentation and discovery within the subject matter.
Learning-oriented	The response indicates that the participant views technology as a facilitator of student learning, focusing on its role in helping students explore and understand mathematical concepts in greater depth.
Accuracy-oriented	The response shows that the participant values technology for its capacity to enhance efficiency, simplify processes, and ensure precision in teaching and learning practices.
Teaching-oriented	The response demonstrates that the participant perceives technology dependence as a potential obstacle and expresses concerns about its impact on the effective teaching of mathematics.

Preliminary analyses included the distribution of SCK, argumentation schedules, and technology perspective among TCs for each task, and Spearman's rank-order correlation coefficient for the scores of three central constructs. Secondly, we explored TC profiles regarding SCK, argumentation schemes, and the technology perspectives they demonstrated for each task. To establish TC profiles, we created a case database for each TC, developed evidence from both Tasks, tabulated the categories of SCK, argumentation schemes, and technological perspectives, and examined repeating patterns among these profiles (Yin, 2009).

Results

This section initially reports quantitative findings, frequency tables for each mathematics task crossed with each level of the three central constructs (Table 2), and Spearman Rho correlations. Secondly, we present *common* TC profiles, incorporating qualitative evidence concerning TC's SCK, argumentation schemes, and technology perspectives from Tasks 1 and 2.

Table 2

Distribution of TCs' levels of SCK, argumentation schemes, and technological perspectives in each Task

	Scores/Labels	Task 1	Task 2
SCK	Robust	8 (53%)	8 (53%)
	Limited	4 (27%)	6 (40%)
	Lack	3 (20%)	1 (7%)
Argumentation Schemes	Analytical	2 (13%)	8 (53%)
	Empirical	10 (67%)	1 (7%)
	Externally-based	3 (20%)	6 (40%)
Technological Perspectives	Experiment-oriented	5 (33%)	3 (20%)
	Learning-oriented	4 (27%)	2 (13%)
	Accuracy-oriented	3 (20%)	3 (20%)
	Teaching-oriented	3 (20%)	7 (47%)

Table 2 demonstrates that half of the participants showed robust SCK in both Tasks, validly evaluating the mathematical arguments of hypothetical secondary students with clear explanations. The distribution does not reveal a particular difference in TCs' SCK between Tasks 1 and 2. On the other hand, Table 2 indicates a contrast within the type of TCs' argumentation scheme they demonstrated between the two tasks: we observe only 2 TCs utilized an analytical argumentation scheme with Task 1, compared to 8 TCs with Task 2. We interpret this finding in two ways: first, the cognitive demand for Task 2 (the law of cosines) was higher than for Task 1 (triangle inequality theorem); the former might have brought a challenge for more TCs to engage in analytical argumentation (i.e., justifications with the use of algebraic/geometric rules). Secondly, the hypothetical student error and mathematical representations demonstrated with Task 1 might have guided students to focus more on numerical examples and empirical evidence than justifications based on mathematical rules and principles.

TCs' responses also revealed a distinction between Tasks 1 and 2 in their technological perspectives. While nine TCs discussed the advantages of using technologies such as GeoGebra to construct, manipulate, and experiment with geometrical representations (i.e., experiment-oriented) and make sense of the geometrical principles (learning-oriented) in Task 1, only five TCs' responses were categorized in these technological perspectives in Task 2. Secondly, more TCs demonstrated a teaching-oriented technology perspective with Task 2 than with Task 1. This difference stems from the representation provided with each task: TCs engaged with *dynamic* GeoGebra representations with Task 1 and *static* imagery representations on paper with Task 2. In that respect, the static representations in Task 2 seem to hold onto teaching-oriented technology perspectives.

Spearman's Rho between TCs' SCK scores and argumentation schemes was +0.603 ($p<0.01$). We found no significant correlations between TCs' technological perspectives and the other two central constructs (i.e., SCK and argumentation schemes). Interpreting this finding, the more TCs utilize *analytical* argumentation schemes, the more likely they will demonstrate *robust* SCK. This finding, we believe, stems from the definition of SCK, which includes "evaluating the plausibility of students' claims (often quickly), giving or evaluating mathematical explanations, [...] selecting representations for particular purposes" (Ball et al., 2008, p. 400).

Two contrasting TC profiles emerged from our qualitative analyses. We recognized that four cases (27%) demonstrated robust SCK, an analytical argumentation scheme, and an experiment-oriented technological perspective. Secondly, we found three cases (20%) representing limited SCK, an externally-based argumentation scheme, and a teaching-oriented technological perspective. In the following two sections, we describe these two contrasting profiles:

TC12: Analytical Argumentation and Experiment-oriented Technological Perspective

For Task 1, TCs were asked to first interpret Marco's error regarding the length of a triangle's side, given the lengths of the other two sides. For this task, TCs were not informed that the task was about the Triangle Inequality Theorem. TCs were guided to play with a GeoGebra applet (i.e., *dynamic* representations) to make sense of Marco's error. TC12 shared the following as their written interpretation of Marco's error:

> "You cannot make a triangle with those [measurements]. As shown in GeoGebra, the 3 and 4 [unit] side lengths are not long enough to meet together since the other length is nine units long and 3+7 is 7 [unit]. For this triangle to work, the 3 and 4 [unit] sides must be made longer to add up to a number greater than 9 [unit]. It cannot be equal to 9 [unit] because connecting those two would have to be right on top of the line, making the whole thing a line and not a triangle. Trying a triangle with measures 4, 5.5, and 9 [unit], for example, as shown in GeoGebra, does form a triangle. As long as the two smallest lengths added together are greater than the third length, I can make a triangle".

In his explanation, TC12 references their experience with the dynamic GeoGebra applet and validly evaluates Marco's argumentation based on those experiences. TC12 applies the Triangle Inequality Theorem without explicitly stating it and explores their experimentation with GeoGebra. As TC12 accurately recognized mathematical ideas under the dynamic GeoGebra representation and linked them to evaluate the plausibility of Marco's claim validly, we categorized their SCK as *robust*.

When asked what the length of a should be equal to in terms of the lengths of b and c for any given ABC triangle, TC12 provided the following for Task 1 and explained his reasoning as follows:

> "If a is greater than or equal to b and c, a triangle can be created if $b + c > a$. If a is less than or equal to b but greater than or equal to c, then a triangle can be created if $a + c > b$. If a is less than or equal to c but greater than or equal to b, a triangle can be created if $a + b > c$. These are all for the same reasons stated above: the two smallest lengths must be added to be greater than the third length. [...] My relationship works because, as I said, for the two shortest sides to be connected, they must be equal to at least the same length as the third side. [...] This can also be seen visually with GeoGebra. They cannot be the same either, since while they can reach each other, they will just form one line and not a triangle".

TC12 is analytical in his argumentation above to justify his reasoning: first, they create three cases based on the length relationship among the triangle's three sides. Then, TC12 explained why their reasoning is valid, referencing the dynamic representations they explored with GeoGebra. Rather than pinpointing a specific example, TC12 justifies their thinking by looking at the general trends they observed with GeoGebra and incorporates them into their justifications. For instance, they explain that the sum of the lengths of two sides cannot be equal to the length of the third side of the triangle, and that such a situation would "form one line and not a triangle".

Finally, for Task 1, when asked to reflect on the impact of GeoGebra representations in his thinking about the Triangle Inequality Theorem, TC12 responded as follows:

> "The applet allowed me to quickly find the yes or no answer to number one [in Task 1]. Without even having to think about it logically, I was able to easily see that it could not be made into a triangle. From there, using the visuals I had, I was able to determine the reason logically and prove it with experimentation and some algebra. I was able to test several examples using the rule I made to see that it worked every time and that when it did not follow my rule, it never created a triangle".

In his response, TC12 describes his experience with DGS, how it can afford test geometry claims, and experiments with it to check the validity of mathematical ideas. In a way, the GeoGebra applet allowed TC12 to bridge his initial empirical argumentation to analytical. We categorized TC12's response as an example of an experiment-oriented technological perspective, which was more common among TCs for Task 1. Interestingly, this pattern in TCs' perspectives contrasted with the technological perspective (i.e., teaching-oriented), which is the most common among TCs for Task 2.

TC15: Externally-based Argumentation and Teaching-oriented Technological Perspective

Task 2 required TCs to understand the reasoning of a high school student, Catherine, about finding the length of a side of a triangle when the lengths of two other sides and the angle measure between these two sides were given. TCs were expected to recognize that Catherine misused the *law of cosines* for the problem. Unlike in Task 1, TCs were tasked in Task 2 with interpreting this law with a sequence of *static* visual representations provided on a Word document, not with DGS. TC15 evaluated the plausibility of Catherine's claims as shown below:

> "Catherine's work is correct; however, she made a mistake when plugging the numbers into the formula. The formula states to add , which is 20° as stated in the problem, but she plugged in , which measures angle C. Thus, she got an incorrect answer. She had the correct procedure, but that mistake carried through her work".

In the second part of Task 1, TCs were asked to interpret the static representations (see appendix) used to visualize the construction of the law of cosines. We expected TCs to recognize the following with these static representations:

- Translation of the base of the initial hatching and gray shaded rectangles (i.e., creating parallelograms having the same base and height) (see the change from Figures 4 to 5, 7 to 8, and 10 to 11 in the appendix),
- Preservation of the area measures between initial rectangles and parallelograms,

- Rotation of the parallelograms around a point (see the change from Figures 5 to 6, 8 to 9, and 11 to 12 in the appendix),
- Translation of the base of the parallelograms (i.e., creating final rectangles having the same base and height) (see the change from Figures 6 to 7, 9 to 10, and 12 to 13 in the appendix),
- Preservation of the area measures between the parallelograms and the final hatching and gray shaded rectangles, and
- Using trigonometric ratios to find one side of the right triangle from its hypotenuse (Figure 14).

Whereas TC15 validly evaluated Catherine's claims, they could not interpret the static representations as we expected. TC15's response for this part of Task 2 is provided below:

> "I assume the colored boxes [in the figures] represent the information we know based on the problem. Our goal [with these figures] is to show how these pieces connect to represent the equation to find the unknown side at the end".

As seen above, TC15's response does not provide an interpretation of the geometrical principles behind the figures. Therefore, we categorized TC15's SCK as limited based on this response. When asked if Catherine's last statement was mathematically correct and how they knew if it was accurate, TC15 provided the following response:

> "The law of cosines is [used] to find the length of a side when the triangle given is not a right angle. In this case, the unknown angle side was *a*; thus, her question is correct because the length of the side you want to find = the sum of the squares of both given sides, minus two times the product of given sides, also times the cosine of the angle you are trying to find the length for. The equation can be moved around depending on what information is given and what you are trying to solve".

In this excerpt, TC15 restates the formula for the law of cosines and explains when it should be used to find the unknown in a related problem. In this statement, TC15 refers to the right triangle as an exception to the law of cosines. This part of their mathematical reasoning is invalid as 90° converts the law of cosines to the Pythagorean theorem, which accurately holds the law for this specific triangle case. As their response includes a superficial reference to the law without providing any justification, and the argument lacks depth of analytical reasoning, we categorized TC15's argumentation scheme as *externally-based*.

To explore TC15's technological perspectives, they were posed to reflect on the impact of various technological representations (e.g., static representations on paper [Appendix] versus *alternative* dynamic representations) in constructing the law of cosines. The following was their written response to this prompt:

> "The [static] figures did not play a role in my justification [of the law of cosines]. I did not understand why this representation best described the reasoning for the law of cosines. When we talk about this law, drawing would require a more in-depth introduction to college-level thinking because it is a form of advanced Geometry/Calculus topic. The figures [in the appendix] almost represent a 3-D model advanced for those learning about the law as freshmen/sophomore [high school students] in Geometry".

In their response, TC15 perceives *static* representations as a potential obstacle and expresses concerns about their impact on the effective teaching of mathematics in high school. Since there is no evidence of alternative technologies considered by TC15 in this excerpt and the emphasis on teaching mathematics, we categorized TC15's technological perspective as *teaching-oriented*. Seven TCs (47%), the majority, demonstrated a teaching-oriented perspective on technology after they engaged with Task 2. This finding stems from TCs' frustration with the static representations provided for Task 2, and they considered that *teaching* the law first in a traditional way (without technology) should be done first to help students understand and then apply it in a problem situation.

Discussions

These preliminary findings indicate some interrelatedness of TCs' SCK and their argumentation schemes. To support TCs' content knowledge development and argumentation skills, dynamic geometry tasks could be designed where TCs are intentionally asked to assess and argue analytically about hypothetical students' reasoning.

Approximately half of the participants demonstrated robust SCK when evaluating student mathematical reasoning. However, a significant portion exhibited limited or lacking SCK, highlighting gaps in their ability to assess mathematical arguments. TCs displayed different levels of argumentation, with empirical argumentation being more prevalent in tasks involving dynamic representations, and analytical argumentation being more common in tasks with static representations. A strong correlation was found between robust SCK and the use of analytical argumentation, emphasizing the importance of structured reasoning in content knowledge development. Furthermore, TCs' views on technology varied depending on the task.

In dynamic geometry environments, TCs tended to adopt experiment-oriented perspectives, valuing technology for exploration. In static representation tasks, they leaned toward teaching-oriented perspectives, seeing technology as a potential obstacle to instruction.

The findings of this study align with and extend prior research on the role of DGS in developing SCK, argumentation skills, and technology perspectives among prospective mathematics teachers. In the study, while half of the TCs demonstrated robust SCK, others exhibited limited or lacking SCK, particularly in evaluating mathematical arguments. This is consistent with Zambak and Tyminski (2017, 2023), who found that the development of SCK with DGS depends on TCs' engagement with technology-rich tasks and their prior technological beliefs. Further, the authors emphasized that TCs with strong SCK tend to recognize multiple representations and their connections, whereas those with weaker SCK struggle to generalize findings beyond empirical observations. This study supports the same claim by showing that TCs who engaged more analytically with DGS-based tasks demonstrated more robust SCK.

Similarly, Segal and associates (2021) found that DGS can enhance SCK, but its impact varies depending on the quality of reflection and structured guidance provided. In this study, TCs engaged more deeply with dynamic representations when prompted to explore mathematical structures actively. However, when exposed to static representations, they struggled to make meaningful connections, mirroring Segal and associates' findings on the necessity of intentional scaffolding.

The relationship between SCK and argumentation in this study closely aligns with Campbell and Zelkowski (2020), who systematically reviewed the literature on how technology supports proof and argumentation. The researchers found that while DGS aids visualization and exploration, its impact on rigorous argumentation depends on whether TCs are explicitly encouraged to transition from empirical justification to formal proofs. Our study reinforces this notion by showing that TCs relied more on empirical argumentation when using DGS interactively, but were more analytical in static representation tasks. This implies that TCs may struggle to apply their DGS-based explorations into formalized reasoning without explicit support. Furthermore, well-designed DGS tasks can serve as a catalyst for argumentation by fostering critical engagement with mathematical properties (Cuesta, 2023). The findings of this study suggest that simply providing DGS is insufficient; structured tasks and reflective discussions are necessary to shift TCs from experimental to analytical reasoning.

In this study, TCs demonstrated experiment-oriented perspectives when using DGS, but shifted to teaching-oriented perspectives in static representation tasks aligned with the research of Santos-Trigo and associates (2021), which shows that teachers' use of technology varies significantly based on their perceptions of its role in mathematical learning. The study by Santos-Trigo et al. also underlines that teachers who view technology as an extension of mathematical reasoning are more likely to use it for exploration and deep conceptual engagement, whereas those who see it as a teaching aid may limit its use to procedural instruction. Concurrently, Zambak and Tyminski (2023) noted that teachers' beliefs about technology shape whether they use it to enhance reasoning or simply improve efficiency. This study supports these findings by showing that TCs' perspectives on technology were context-dependent, influenced by the nature of the mathematical tasks presented. Overall, the findings support existing research indicating that DGS can be a powerful tool for enhancing SCK and argumentation but requires intentional instructional design to maximize its impact.

Our findings have some implications for mathematics teachers and teacher educators. We believe mathematics educators should prioritize integrating dynamic geometry software (DGS), such as GeoGebra, within classroom settings when applicable. The use of such tools has the potential to significantly enhance students' conceptual understanding by providing opportunities for interactive exploration and testing of mathematical ideas. Additionally, educators are encouraged to foster analytical argumentation skills in students, moving beyond a reliance on empirical observations to develop more profound, more rigorous mathematical reasoning. Furthermore, our study highlights the need for adjustments in teacher preparation programs. By incorporating DGS-based tasks into teacher education curricula, mathematics teacher educators can better equip teacher candidates (TCs) with the skills necessary to evaluate and engage with mathematical reasoning critically. The inclusion of explicit instruction on argumentation frameworks, such as Toulmin's model, is recommended to strengthen TCs' ability to justify mathematical claims with precision and rigor, thereby enhancing their overall pedagogical competence in fostering critical mathematical thinking among students.

Conclusion

TCs need exposure to solving problems with GeoGebra, constructing geometric figures, examining hypotheses, and exploring hypothetical students' work with GeoGebra. By engaging in these activities, candidates not only strengthened their mathematical reasoning but also critically analyzed the role of technology in facilitating visualization and understanding. Our results highlight the need to integrate cognitively demanding DGS-integrated tasks in work with TCs to provide them with experiences interpreting students' reasoning in technology-oriented learning environments and critically reflect on the role of technology-based tools in mathematics teaching and learning. The study's limitations were the low number of participants and the focus of the mathematics topics covered in each task. Accordingly, we recommend future research to examine these three constructs, SCK, argumentation schemes, and technological perspectives, with a larger sample size using tasks focusing on different mathematics topics (e.g., algebra, functions, number systems). Secondly, we believe future research should explore the perspectives of mathematics teacher educators about the ways they prepare TCs for teaching mathematics with the help of various cognitive tools and representations, and what teacher education practices are more fruitful than others in supporting TCs' meaningful experiences with DGS.

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Appendix

Assume that Catherine is a student in your future mathematics classroom. As their mathematics teacher, you want to help the other students in your classroom understand why Catherine's statement is correct for any given triangle (or you wish to correct her understanding if you think her statement is incorrect). You consider the following visuals (from Figure 3 to Figure 14 with written explanation) *might* help your students understand it. Explore each Figure and the written explanations in the following order:

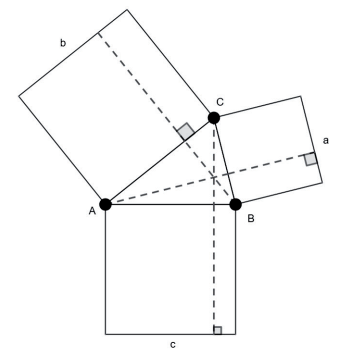
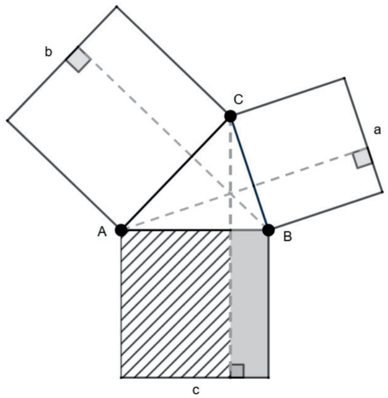
Figure 3	Figure 4
	

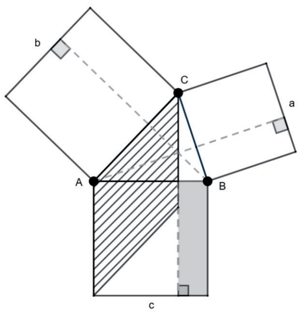
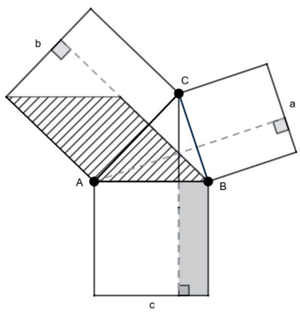
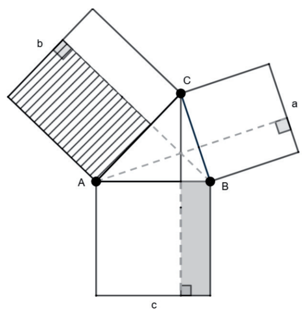
Figure 5	Figure 6	Figure 7
		

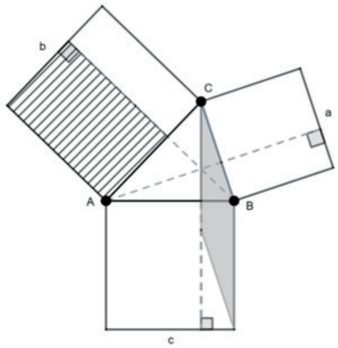
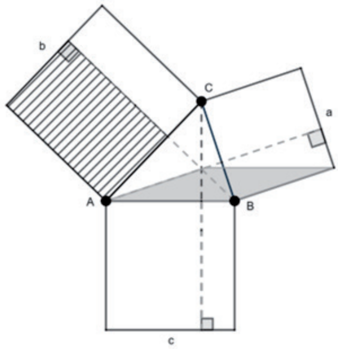
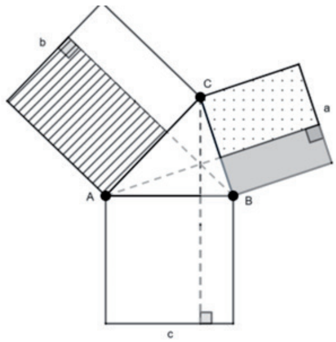
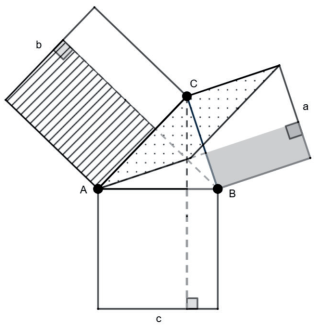
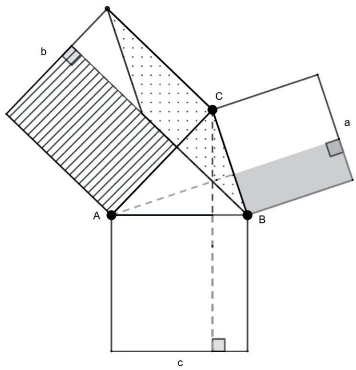
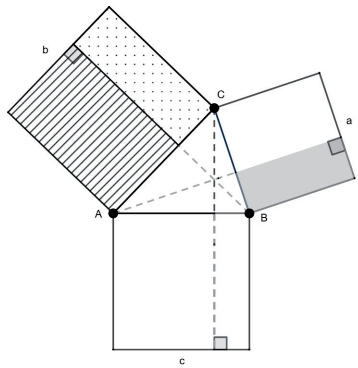
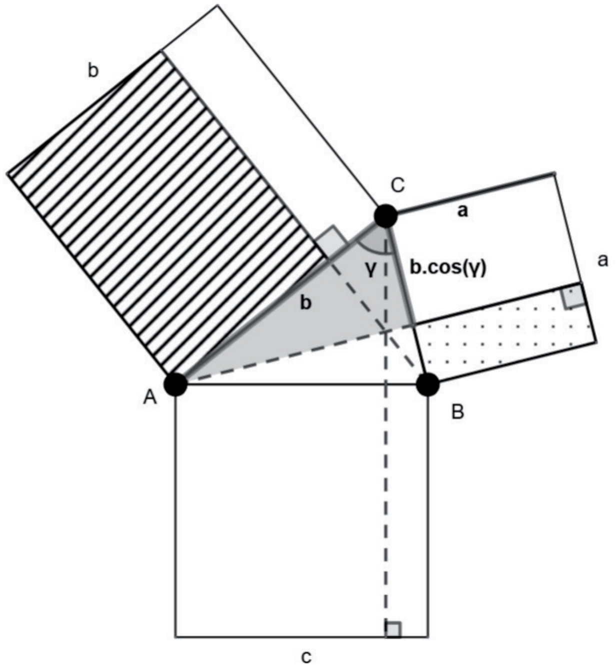
FIGURE 8	FIGURE 9	FIGURE 10
		

FIGURE 11	FIGURE 12	FIGURE 13
		

$c^2 = a^2 + b^2 - c^2 = a^2 + b^2 -$ [Area of the dotted rectangles in Figures 10 and 13]

To find the extra area:

FIGURE 14



Therefore, $c^2 = a^2 + b^2 - 2ab\, cos(\gamma) c^2 = a^2 + b^2 - 2ab\, cos(\gamma)$